On Endogenous Growth with Physical and Human Capital

Jordi Caballé

Universitat Autonoma de Barcelona

Manuel S. Santos

Universidad Carlos III de Madrid

This paper presents a class of models in which agents may devote part of their nonleisure activities to going to school so as to increase the efficiency units of labor they supply to the firms and the wages they receive. The interaction among the technology of human capital accumulation and agents' preferences will determine endogenously the economy's rate of growth. Given a constant returns to scale technology for physical capital accumulation, we characterize the set of steady states as a ray from the origin and show the global convergence of every off-balanced path to some point on this ray. Further properties concerning the dynamic evolution of the state and control variables around the ray of steady states are also established. Our analysis is useful to understand the role played by the technologies of physical and human capital in the process of accumulation and to evaluate the impact of policies geared toward attaining higher levels of capital. Our results highlight the importance of human capital in the dynamics of growth.

I. Introduction

This paper presents a class of models in which labor may be reproduced in an unbounded fashion through schooling. Agents may de-

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vote part of their nonleisure time to going to school so as to increase the efficiency units of labor they supply in the labor market. The production of the physical good is given by a constant returns to scale technology that depends on the physical capital accumulated and the efficiency units of labor. Very similar models can be found in Uzawa (1965) and Lucas (1988). In this class of models, the interaction among the technologies that allow for the accumulation of physical and human capital and consumers' preferences determines endogenously the economy's rate of growth. Our work complements Uzawa's paper by analyzing the dynamics for the case of a decreasing marginal utility of consumption. Since the objective function in Uzawa (1965) was the discounted sum of consumption, optimal paths displayed periods of either zero consumption or zero physical investment, which greatly simplified the analysis of the dynamics of off-balanced paths. Such types of counterfactual "bang-bang" solutions are ruled out by the assumptions of our model.

Our main departure from Lucas (1988) lies in the modeling of the production sector. We consider general forms of linearly homogeneous production functions for both the consumption good and the educational sector, whereas Lucas considers a Cobb-Douglas production function with external effects in the production of the physical good. In our case, the set of steady states (or balanced paths) is a ray emanating from the origin. We shall establish the global convergence of every off-balanced path to some point on this ray. Our method of proof combines techniques from dynamic programming and the maximum principle. We exploit the fact that the derivatives of the value function define the costate variables of the canonical Hamiltonian equations.

The multiplicity of steady states implies that there are economies with different initial levels of human and physical capital that may end up growing at a common rate, although the long-run levels of those two kinds of capital will never converge. Similar results are obtained in models with externalities (Romer 1986; Lucas 1988) or in the *Ak* model of Barro (1990) and Rebelo (1991).

The global convergence of off-balanced paths makes our model distinctive from the standpoint of the neoclassical two-sector growth model (see Benhabib and Nishimura 1985; Boldrin 1989). In a similar model, the global convergence of optimal paths was originally conjectured by Lucas (1988, p. 25). However, even though in recent years the literature on endogenous growth has also grown at an increasing rate, the dynamics of models with several factors of production are not yet well understood. Progress in this direction should be useful to determine the role played by the technologies of physical and human capital in the process of growth and to evaluate the impact of policies geared toward attaining higher levels of accumulation. These issues arise naturally in our simple model, where for example there is an asymmetric response of consumption to increases in the proportions of physical and human investment.

If the economy is endowed in relative terms with a greater amount of physical capital, then we show (making an appropriate normalization of the levels) that consumption is initially high and decreases along the optimal path toward a given steady-state solution. Such a steady state exhibits a lower amount of physical capital. Conversely, if the economy is endowed with a greater amount of human capital, then the level of consumption is initially low, and it increases toward a given steady-state solution. Such a steady state exhibits a higher amount of physical capital.

In the standard one-sector model (cf. Cass 1965), increases in physical capital do not affect the level of human capital accumulated, which is exogenously given. In this latter model, all optimal orbits converge to a unique level of capital accumulation. In our endogenous growth framework, an increase in physical capital may affect the time devoted to education and thus may induce changes in the amount of human capital accumulated in the economy. Since the level of human capital affects the value of the marginal productivities, a change in physical capital may move the economy to a different steady state. Indeed, we find that an increment in physical capital from a given steady-state solution can lead to the following three situations: (*a*) the *normal* case: the level of human capital goes up and the economy converges toward another steady state with a higher level of physical capital; (*b*) the *paradoxical* case: the level of human capital goes down and the economy converges toward another steady state with a lower level of physical capital; and (*c*) the *exogenous growth* case: the level of human capital remains constant and the economy converges back toward the initial steady state.

There are two countervailing forces that give rise to these three cases. On one hand, a sudden increase in physical capital makes labor more productive, thereby raising the opportunity cost of going to school and thus discouraging the accumulation of human capital. This negative effect is determined by the elasticity of marginal productivity of labor with respect to capital. Also, a sudden increase in physical capital lowers the rate of growth in consumption and in physical capital. The lower rate of growth in consumption encourages human capital accumulation. This positive effect is inversely related to the intertemporal elasticity of substitution: Human capital investment will be less attractive if agents are more willing to intertemporally substitute consumption.

Our analysis thus illustrates that the standard neoclassical model is a special case of the Lucas-Uzawa framework. Our numerical computations will show that for plausible parameter values, endogenous and exogenous growth models generate fairly similar dynamics around a given steady-state solution. In a later stage of our paper, we investigate whether these results are robust to further variations of the original setting. One somewhat more realistic assumption of the Lucas-Uzawa framework is to let physical capital be an input of the educational sector. This case has been initially considered by Mulligan and Sala-i-Martin (1991) and Rebelo and Stokey (1991). We prove in this setting that if the educational sector is intensive in physical capital, then an increase in physical capital will always bring about an increase in the level of education. If, however, the educational sector is intensive in human capital, then it is possible to obtain again the paradoxical and exogenous growth cases. As a consequence, a sudden increase in physical capital may lead the economy to a lower steady state.

Before closing this section, we would like to mention some recent work that has independently analyzed transitional dynamics in similar models of endogenous growth. Faig (1992) considers a simpler version of this model and studies the impact of shocks to technology and public consumption. Chamley (1993) analyzes transitional dynamics in a setting closer to ours. He also includes taxes and externalities, but he does not include the case in which physical capital is an input of the human capital technology. He presents an alternative methodology for the study of off-balanced paths, obtaining similar results. His analysis is, nevertheless, complementary to ours as each approach focuses on certain aspects of the dynamical problem.

The paper is organized as follows. Section II presents the model. Section III discusses the necessary and sufficient conditions for the existence of a balanced competitive equilibrium and some properties of the equilibrium. The problem of global stability to the set of balanced steady states is analyzed in Section IV. The normal, paradoxical, and exogenous growth cases are characterized in Section V. The case in which physical capital enters as an input of the educational sector is discussed in Section VI. We conclude in Section VII with a summary of our main findings.

II. The Model

We consider a standard optimal growth model with an unbounded horizon. The economy consists of households (or infinitely lived, growing dynasties) and competitive firms. In the absence of externalities, the solution of the problem selected by a planner coincides with a competitive solution achieved in a decentralized manner through competition among firms and optimizing behavior of the dynasties. Thus we limit our analysis to the planner's problem without loss of generality.

Each individual of the economy derives utility from consuming $\tilde{c}(t)$ units of a single good at each moment in time. Preferences are characterized by an instantaneous C^2 utility function $U(\tilde{c}(t))$ with $U'(\tilde{c}) > 0$ and $U''(\tilde{c}) < 0$ for all $\tilde{c} > 0$, $\lim_{\tilde{c}\to 0} U'(\tilde{c}) = \infty$, and $\lim_{\tilde{c}\to\infty} U'(\tilde{c}) = 0$.[1] The number of individuals in each dynasty is $N(t)$, and the exogenous instantaneous rate of population growth is n. Each dynasty discounts the utility of future consumption of each member of the household at the rate ρ.

The technology used by the firms is represented by a C^2 production function $F(\tilde{K}(t), \tilde{L}(t))$. This function is concave, increasingly monotone, and linearly homogeneous on the capital $\tilde{K}(t)$ and the efficiency units of labor $\tilde{L}(t)$ used by each firm. To guarantee certain interiority conditions, this function exhibits unbounded partial derivatives at the boundary, and both factors are essential in the production process. More precisely, for each $\overline{K} > 0$ and $\overline{L} > 0$,

$$
\begin{aligned}
\lim_{\tilde{L}\to 0} F_L(\overline{K}, \tilde{L}) &= \infty, \\
\lim_{\tilde{K}\to 0} F_K(\tilde{K}, \overline{L}) &= \infty, \qquad (1)\\
F(0, \overline{L}) = 0, \quad F(\overline{K}, 0) &= 0,
\end{aligned}
$$

where the subindex denotes the variable with respect to which the partial derivative is taken, and $\overline{K}$ and $\overline{L}$ remain fixed. Also, $F_{KK}(\tilde{K}, \tilde{L}) < 0$ and $F_{LL}(\tilde{K}, \tilde{L}) < 0$. Since firms are assumed to behave competitively, wages are equal to the marginal productivity of each efficiency unit of labor supplied by a worker. Individuals also receive a return from their savings that is equal to the marginal productivity of capital. The single good produced by each firm may be either used as a consumption good or invested as physical capital.

Each individual owns one unit of nonleisure time per period. If a worker devotes the fraction $u(t)$ of his or her nonleisure time to work and the efficiency per unit of labor supplied is $\tilde{h}(t)$, then $\tilde{L}(t) = N(t)u(t)\tilde{h}(t)$. The remaining $1 - u(t)$ of the nonleisure time is devoted to accumulating human capital through schooling. The technology of human capital growth we postulate is $\dot{\tilde{h}}(t) = \tilde{G}(\tilde{h}(t), 1 - u(t))$, where $\tilde{G}(\cdot, \cdot)$ is a C^2 production function such that, for each fixed $\overline{h} > 0$, the mapping $\tilde{G}(\overline{h}, \cdot)$ is concave and has a positive derivative.

[1] We say that a function is C^2 if it is continuous over the domain of definition and has continuous partial, second-order derivatives at each interior point.

Both physical and human capital depreciate at constant rates, which are $\pi \geq 0$ and $\theta \geq 0$, respectively.

Given the equivalence between the centralized and the competitive solution, we shall limit our analysis to the optimal problem.

DEFINITION 1. An optimal solution for the economy described above is a set of paths $\{\tilde{c}(t), \tilde{K}(t), \tilde{h}(t), u(t)\}$ that solve the following optimization problem:

$$\max \int_0^\infty U(\tilde{c}(t))N(t)e^{-\rho t}dt \tag{P}$$

subject to

$$\dot{\tilde{K}}(t) = F(\tilde{K}(t), N(t)u(t)\tilde{h}(t)) - \pi\tilde{K}(t) - N(t)\tilde{c}(t), \tag{2}$$

$$\dot{\tilde{h}}(t) = \tilde{G}(\tilde{h}(t), 1 - u(t)) - \theta\tilde{h}(t), \tag{3}$$

$$\tilde{K}(0) = K_0, \quad \tilde{h}(0) = h_0, \quad N(0) = N_0, \tag{4}$$

$$\tilde{c}(t) \geq 0, \quad u(t) \in [0, 1], \quad \tilde{K}(t) \geq 0, \quad \tilde{h}(t) \geq 0. \tag{5}$$

Observe that equations (2) and (3) are the resource constraints faced by the economy in question, and (4) gives the initial conditions of the state variables, $\tilde{K}(t)$ and $\tilde{h}(t)$, and population $N(t)$.

DEFINITION 2. A balanced optimal path (or steady-state equilibrium) is an optimal solution $\{\tilde{c}(t), \tilde{K}(t), \tilde{h}(t), u(t)\}$ to the optimization problem (P) for some initial conditions $\tilde{K}(0) = K_0$ and $\tilde{h}(0) = h_0$, such that $\tilde{c}(t)$, $\tilde{K}(t)$, and $\tilde{h}(t)$ grow at constant rates, $u(t)$ is constant, and the output/capital ratio is constant.

Since from casual empiricism we observe a positive investment in human capital, we shall focus on balanced equilibria with $u(t) = u^* < 1$. The next section will discuss some properties of an interior balanced path and the conditions for its existence.

III. Existence and Properties of a Balanced Path

This section lays down the properties of a balanced steady state and the conditions for its existence. A more detailed analysis can be found in King, Plosser, and Rebelo (1988) and Mulligan and Sala-i-Martin (1991).

Denote by ν the rate of growth of human capital in a balanced equilibrium. Since the output/capital ratio is constant in such an equilibrium (and given the linear homogeneity of the production function), the rate of growth of $\tilde{K}(t)$ in a balanced equilibrium must therefore be the same as the one of $\tilde{L}(t) = N(t)u(t)\tilde{h}(t)$, which is equal to $\nu + n$. Furthermore, dividing (2) by $\tilde{K}(t)$, we see that $\tilde{K}(t)$ must grow at the same rate as $N(t)\tilde{c}(t)$, which implies that the rate of growth of per capita consumption is also equal to ν. Finally, dividing (3) by $\tilde{h}(t)$, we see that a balanced competitive equilibrium is compatible only with a specification of the technology for human capital accumulation that is linearly homogeneous on $\tilde{h}(t)$. That is,

$$\tilde{G}(\tilde{h}(t), 1 - u(t)) = \tilde{h}(t)G(1 - u(t)), \tag{6}$$

with $G' > 0$ and $G'' \leq 0$.

In order to proceed with our analysis of the set of steady states, we now normalize the variables $\tilde{c}(t)$, $\tilde{K}(t)$, and $\tilde{h}(t)$ as follows:

$$c(t) = \tilde{c}(t)e^{-\nu t}, \tag{7}$$

$$K(t) = \tilde{K}(t)e^{-(\nu+n)t}, \tag{8}$$

$$h(t) = \tilde{h}(t)e^{-\nu t}. \tag{9}$$

Notice that in this redefinition of the variables $\tilde{c}(t)$, $\tilde{K}(t)$, and $\tilde{h}(t)$, we have used their respective rates of growth in a balanced equilibrium as the discounting parameter. Hence, the normalized variables $c(t)$, $K(t)$, and $h(t)$ will remain constant along a balanced path, and c^*, K^*, and h^* will denote the respective steady-state values of these variables.

In view of (7)–(9), constraint (2) becomes

$$\dot{K}(t) = F(K(t), N_0u(t)h(t)) - (\nu + n + \pi)K(t) - N_0c(t). \tag{10}$$

Also, from (6), constraint (3) becomes

$$\dot{h}(t) = h(t)[G(1 - u(t)) - (\nu + \theta)]. \tag{11}$$

We now summarize some relevant facts of a steady state in the following two propositions. The first one refers to necessary and sufficient conditions for the existence of an interior balanced equilibrium, and the second one refers to some properties of such a balanced equilibrium.

PROPOSITION 1. Consider the dynamic optimization problem (P), where $F(\cdot, \cdot)$ is a C^2 linearly homogeneous, strictly increasing, and concave function that satisfies condition (1), and $F_{KK}(\tilde{K}, \tilde{L}) < 0$, $F_{LL}(\tilde{K}, \tilde{L}) < 0$; $U(\cdot)$ is a C^2 function with $U'(\tilde{c}) > 0$ and $U''(\tilde{c}) < 0$ for all $\tilde{c} > 0$; and $\tilde{G}(\cdot, \cdot)$ is a C^2 function such that, for each fixed $\bar{h} > 0$, the mapping $\tilde{G}(\bar{h}, \cdot)$ is concave and has a positive derivative. Assume that U and $\tilde{G}$ satisfy

$$\rho + \pi - \frac{U''(\bar{c})\bar{c}}{U'(\bar{c})}\left[\frac{\tilde{G}(\bar{h}, 0)}{\bar{h}} - \theta\right] > 0 \quad \text{for all fixed } \bar{c}, \bar{h} \geq 0. \tag{12}$$

The following conditions are necessary and sufficient for the existence of an interior balanced path:

a) The utility function $U(\cdot)$ must exhibit a constant elasticity of intertemporal substitution $\sigma^{-1} > 0$.

b) The technology for human capital production $\tilde{G}(\cdot, \cdot)$ must take the functional form

$$\tilde{G}(\tilde{h}, 1 - u(t)) = \tilde{h}(t)G(1 - u(t)) \quad \text{with } G' > 0, G'' \leq 0,$$

$$(1 - \sigma)G(1) < \rho - n + (1 - \sigma)\theta < (1 - \sigma)G(0) + G'(0). \quad (13)$$

PROPOSITION 2. Consider the dynamic optimization problem (P) with the same assumptions as in proposition 1. If u^* is the fraction of time devoted to work and ν is the rate of growth of human capital in a balanced interior path, then

a) u^* is the unique solution to

$$\rho - n = G'(1 - u^*)u^* + (1 - \sigma)[G(1 - u^*) - \theta], \quad (14)$$

and $\nu = G(1 - u^*) - \theta$;

b) the common steady-state rate of growth of consumption per capita $\tilde{c}(t)$ and human capital $\tilde{h}(t)$ must be ν, and the rate of growth of physical capital $\tilde{K}(t)$ must be $\nu + n$; and

c) the set of steady-state values of $c(t)$, $K(t)$, and $h(t)$ is a linear manifold of dimension one emanating from the origin.

Part *a* of proposition 1 states that in a balanced path $\{c^*, K^*, h^*, u^*\}$ the elasticity of intertemporal substitution must remain constant. Observe that in a balanced path the output/capital ratio is constant, and thus $F_K(K^*, N_0u^*h^*)$ must be constant. Moreover, it is straightforward to show that utility maximization implies that

$$F_K(K^*, N_0u^*h^*) - \pi = \rho - \frac{d}{dt}\log U'(e^{\nu t}c^*), \quad (15)$$

where $d[\log U'(e^{\nu t}c^*)]/dt = -\sigma(t)\nu$, and $\sigma(t) = -U''(e^{\nu t}c^*)e^{\nu t}c^* \div U'(e^{\nu t}c^*)$. Hence, from (15) we can see that the elasticity of intertemporal substitution $\sigma(t)^{-1}$ must be constant along a balanced path, that is,

$$F_K(K^*, N_0u^*h^*) - \pi = \rho + \sigma\nu. \quad (16)$$

The utility function $U(c)$ must therefore take the functional form $A + B[c^{1-\sigma}/(1 - \sigma)]$, with $B > 0$ and $\sigma > 0$.

The first inequality in (13) plays the role of a transversality condition, and it is analogous to condition (12) in Uzawa (1965). If $(1 - \sigma)G(1) > \rho - n + (1 - \sigma)\theta$, then the objective in (P) may take on an unbounded value. Conversely, if $G(0)$ and $G'(0)$ are too low, then investment in human capital is not profitable; thus an interior balanced path is not achieved. For the technology of human capital growth analyzed by Lucas (1988), $G(1 - u) = \delta(1 - u)$ with $\delta > 0$, condition (13) implies that $(1 - \sigma)\delta < \rho - n + (1 - \sigma)\theta < \delta$. Equation (14) is the analogue of condition (16) for the technology of human capital accumulation. The existence of a unique u^* comes from (13) and the concavity of G. Observe from (14) that the steady-state values u^* and ν are fully determined by the specification of the human capital technology and agents' preferences. This is to be contrasted with the case of increasing returns to scale (Lucas 1988) in which the size of the externality for producing the physical good also matters in the determination of the variables u^* and ν. The existence of a steady-state pair (K^*, h^*) to solve for (16) is ensured by conditions (1) and (12). Note that (12) is simply a mild bound on the productivity of human capital. Finally, the assumption of constant returns to scale for both F and G and the constant elasticity of substitution utility function imply that the set of steady states is a ray from the origin (part c of proposition 2). That is, economies with twice as much stock of both K^* and h^* will consume twice as much and will devote an identical fraction to working time, u^*.

IV. Dynamics

To study the dynamics of this model, we rewrite our optimization problem, as dictated by proposition 1, and define the value function as

$$V(K_0, h_0) = \max \int_0^\infty N_0 \frac{[c(t)]^{1-\sigma}}{1 - \sigma} e^{-[\rho - n - (1-\sigma)\nu]t} dt \tag{P'}$$

subject to

$$\dot{K}(t) = F(K(t), N_0 u(t) h(t)) - (\nu + n + \pi)K(t) - N_0 c(t) \tag{10}$$

and

$$\dot{h}(t) = h(t)[G(1 - u(t)) - (\nu + \theta)], \tag{11}$$

$$K(0) = K_0, \quad h(0) = h_0, \quad c(t) \geq 0, \quad u(t) \in [0, 1], \quad h(t) \geq 0, \quad K(t) \geq 0,$$

where $\rho - n + (1 - \sigma)\nu > 0$ (see [14]). We also write the corresponding Hamiltonian function:

$$\begin{aligned} H(K(t), h(t), c(t), u(t), \gamma_1(t), \gamma_2(t), t) = e^{-[\rho - n - (1-\sigma)\nu]t} \Big(N_0 \frac{[c(t)]^{1-\sigma}}{1 - \sigma} \\ + \gamma_1(t)[F(K(t), N_0 u(t) h(t)) - (\nu + n + \pi)K(t) - N_0 c(t)] \\ + \gamma_2(t)\{h(t)[G(1 - u(t)) - (\nu + \theta)]\} \Big). \end{aligned} \tag{17}$$

It follows from the maximum principle that along the optimal path $\{c(t), K(t), h(t), u(t)\}$ the law of motion of the costate variables is characterized by the following equations:

$$\frac{\dot{\gamma}_1(t)}{\gamma_1(t)} = \rho + \sigma\nu - F_K(K(t), N_0 u(t) h(t)) + \pi \quad (18)$$

and

$$\frac{\dot{\gamma}_2(t)}{\gamma_2(t)} = \rho - n + \sigma\nu - G'(1 - u(t))u(t) - G(1 - u(t)) + \theta. \quad (19)$$

Assuming the interiority of optimal paths $\{c(t), K(t), h(t), u(t)\}$, we know from Santos (1990) that the value function V is a C^2 concave mapping. Furthermore, it is also known (cf. Benveniste and Scheinkman 1982) that the first-order derivative $DV(K_0, h_0) = (\gamma_1(0), \gamma_2(0))$, where $\gamma_1(0)$ and $\gamma_2(0)$ are the values of the costate variables for (17) at time 0.[2]

Note that in problem (P′) the instantaneous utility function is isoelastic, and the feasibility constraints (10) and (11) are linearly homogeneous in the variables $c(t)$, $K(t)$, and $h(t)$. This implies that the policy function $g(K_0, h_0)$ is homogeneous of degree one, and the value function $V(K_0, h_0)$ is homogeneous of degree $1 - \sigma$. Also, as concluded in the previous section, all stationary points are located on a ray of physical and human capital pairs.

We next show that all other interior initial conditions converge to some point on the ray in a particular fashion. To illustrate our method of proof, we consider an arbitrary point, say point a, where the ratio K/h is higher than that of a stationary point. Let us distinguish the following cases (see fig. 1).

Case A.—The vector field points toward the upper right-hand-side quadrant. In this case, both human and physical capital are increasing. Human capital can increase only if $u(t) < u^*$. It then follows from equation (19) that $\dot{\gamma}_2(t) \geq 0$, since the derivative of (19) with respect to $u(t)$ is $G''(1 - u(t))u(t) \leq 0$. Furthermore, since $K/h > K^*/h^*$ and $u(t) < u^*$, it must be the case (eq. [18]) that $\dot{\gamma}_1(t) > 0$. We may deduce that there is a particular point such as b, in figure 2, where the corresponding costate variables, γ_1^b and γ_2^b, are such that $\gamma_1^b > \gamma_1^a$ and $\gamma_2^b \geq \gamma_2^a$. Consider now point c, on the same ray as a, and with the property that $K_b = K_c$. Given that the value function is

[2] Our analysis is readily extended to the case of noninterior optimal paths (i.e., $u(t) = 1$ for some t in $[0, \infty)$), since the results of Benveniste and Scheinkman apply to solutions at the boundary. It should be observed that, in the context of our model, at each point there is only a unique pair of costate variables that satisfy the conditions of Benveniste and Scheinkman (1982, theorem 2).

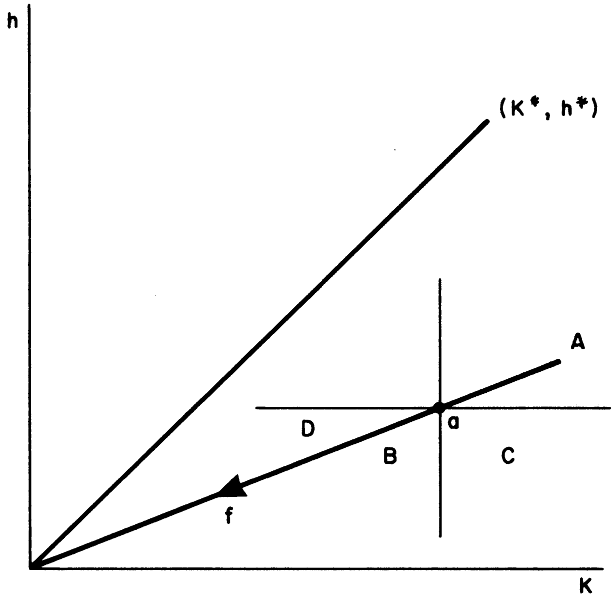

FIG. 1

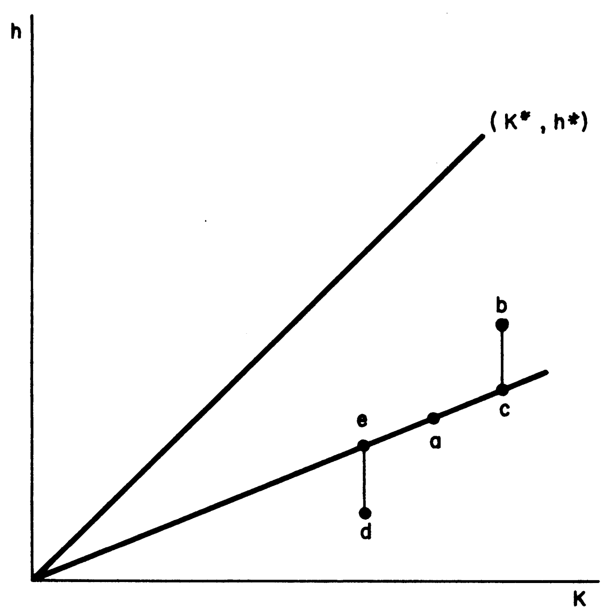

FIG. 2

homogeneous of degree $1 - \sigma < 1$, it must be true that $\gamma_2^a > \gamma_2^c$. Moreover, by the strict concavity of V, it must hold that $\gamma_2^c > \gamma_2^b$. Consequently, $\gamma_2^a > \gamma_2^b$, which is a contradiction. A similar contradictory argument can be applied to γ_1 if b is below the ray passing through a. Therefore, the vector field generated by the policy function $g(K, h)$ cannot point toward the upper right-hand-side quadrant.

Case B.—The vector field points toward area B. In this case, both physical and human capital are decreasing. Therefore, $u(t) > u^*$. It then follows from equation (19) that $\dot{\gamma}_2(t) \leq 0$. Suppose thus that our particular point is d of figure 2. Then $\gamma_2^a \geq \gamma_2^d$. However, this is impossible since $\gamma_2^d > \gamma_2^e$ and $\gamma_2^e > \gamma_2^a$.

Case C.—The vector field points toward the lower right-hand-side quadrant. In view of cases A and B, this situation can be ruled out, since otherwise it would imply an overaccumulation of physical capital with asymptotically zero consumption. That is, given $\gamma_1(t) = c(t)^{-\sigma}$, one can see from (18) that $c(t)$ converges exponentially to zero, as $K(t)/h(t)$ gets unbounded. It is easy to see that this situation is not optimal.

Observe that we have not ruled out vector fields pointing toward area D of figure 1. However, the optimal policy cannot point toward the direction $\vec{af}$, since by equation (19) $\dot{\gamma}_2(t) \leq 0$, and moving along $\vec{af}$ implies $\dot{\gamma}_2(t) > 0$. Therefore, the vector field generated by the optimal policy at each point lies separated from the direction emanating from the origin. Moreover, concerning case A, the concavity of V is enough to rule out the situation in which the vector field is a vertical vector. The shaded area in figure 3 illustrates then the set of possible directions corresponding to the vector field generated by the optimal policy function near point a. In figure 3, we have also drawn the set of possible directions corresponding to a point a', with a higher proportion of human capital. In this situation, similar arguments allow us to rule out the corresponding analogues of cases A, B, and C.

Pick now an interior point (K, h). Let $K/h > K'/h' > K^*/h^*$. Since $[K/h, K'/h']$ is a compact interval, the arguments above and the continuity of the optimal policy g imply that an optimal orbit starting at (K, h) must cross the ray $\{(\lambda K', \lambda h'), \lambda > 0\}$. This proves convergence of the ratio K/h to K^*/h^*. Furthermore, as shown below, every interior steady state (K^*, h^*) locally contains a one-dimensional stable manifold transverse to the ray $\{(\lambda K^*, \lambda h^*), \lambda > 0\}$. Hence, the point (K^*, h^*) must attract an interior point of every nearby ray $\{(\lambda K', \lambda h'), \lambda > 0\}$. By the homogeneity of the policy function g, it follows that every interior point of the ray $\{(\lambda K', \lambda h'), \lambda > 0\}$ converges to an interior point of the ray $\{(\lambda K^*, \lambda h^*), \lambda > 0\}$. Consequently, the point (K, h) converges to an interior steady state (K^*, h^*), and (K, h) is an

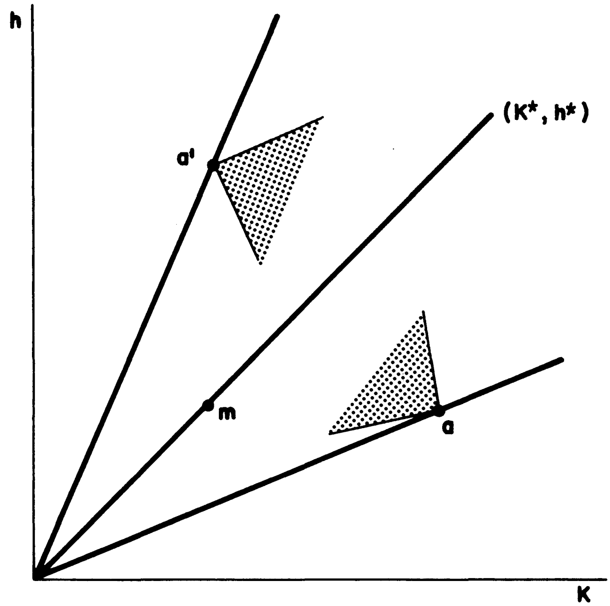

FIG. 3

arbitrary interior point. This proves global stability provided that the system is locally stable.[3]

In order to discuss local stability, we first derive (as in the Appendix) the law of motion of the variables $\{c, K, h, u\}$ by applying the maximum principle to our optimization problem. After elimination of the costate variables, the dynamical system to be considered is composed of the following equations:

$$\dot{c}(t) = -\frac{c(t)}{\sigma}[\rho + \sigma\nu - F_K(K(t), N_0u(t)h(t)) + \pi], \tag{20}$$

[3] After this paper was circulated, Bob Lucas and Mike Woodford pointed out an alternative approach for proving global convergence to the ratio K^*/h^*. Their approach exploits more explicitly the homogeneity of the policy and value functions. Define $z = c/h$ and $x = K/h$. Then problem (P′) may be written as

$$\max \int_0^\infty \frac{[z(t)]^{1-\sigma}}{1-\sigma} h_0^{1-\sigma} e^{(1-\sigma)\int_0^t [G(1-u(s))]ds} N_0 e^{-(\rho-n)t} dt$$

subject to

$$\dot{x}(t) = F(x(t), N_0u(t)) - [n + \pi + G(1 - u(t)) - \theta]x(t) - N_0z(t).$$

This formulation embodies a unique state variable, the ratio $x = K/h$, and it is readily shown that every $x > 0$ converges to the steady-state ratio, x^*. We have not adopted this somewhat simpler framework since an important part of our analysis is concerned with the evolution of the levels K and h.

$$\dot{K}(t) = F(K(t), N_0 u(t)h(t)) - (\nu + n + \pi)K(t) - N_0 c(t), \quad (21)$$

$$\dot{h}(t) = h(t)[G(1 - u(t)) - (\nu + \theta)], \quad (22)$$

$$\dot{u}(t) = \frac{1}{\dfrac{\beta(t)}{u(t)} + \dfrac{\phi(t)}{1 - u(t)}} \Bigg\{ [1 - \beta(t)](n + \pi - \theta) - F_K(K(t), N_0 u(t)h(t))$$
$$+ \beta(t)\frac{F(K(t), N_0 u(t)h(t))}{K(t)} - \beta(t)\frac{N_0 c(t)}{K(t)} \quad (23)$$
$$+ [1 - \beta(t)]G(1 - u(t)) + G'(1 - u(t))u(t) \Bigg\},$$

where $\beta(t)$ and $\phi(t)$ are the elasticities of the marginal productivities,

$$\beta(t) = \frac{F_{LK}(K(t), L(t))K(t)}{F_L(K(t), L(t))} > 0,$$

and

$$\phi(t) = -\frac{G''(1 - u(t))[1 - u(t)]}{G'(1 - u(t))} \geq 0.$$

Observe that only the family of Cobb-Douglas functions exhibits constant elasticities of the marginal productivities. The coefficients matrix of the linearized system around a steady state (c^*, K^*, h^*, u^*) is

$$M(c^*, K^*, h^*, u^*) =$$

$$\begin{pmatrix} 0 & \dfrac{c^*}{\sigma}F_{KK}(K^*, N_0 u^* h^*) & \dfrac{c^*}{\sigma}N_0 u^* F_{KL}(K^*, N_0 u^* h^*) & \dfrac{c^*}{\sigma}N_0 h^* F_{KL}(K^*, N_0 u^* h^*) \\ -N_0 & \rho - n - (1 - \sigma)\nu & N_0 u^* F_L(K^*, N_0 u^* h^*) & N_0 h^* F_L(K^*, N_0 u^* h^*) \\ 0 & 0 & 0 & -h^* G'(1 - u^*) \\ * & * & * & * \end{pmatrix}.$$

We have not written down the last row entries since they will not be relevant for our analysis. The signs of the eigenvalues of this matrix determine the local dynamics of the system around the steady state. These eigenvalues λ_i (i = 1, 2, 3, 4) are the solutions of the characteristic polynomial associated with the matrix $M(c^*, K^*, h^*, u^*)$. Since the set of steady states is a manifold of dimension one, we know that at least one eigenvalue must be zero. This can also be seen by noticing that the vector (c^*, K^*, h^*, 0) belongs to the null space of the matrix $M(c^*, K^*, h^*, u^*)$. It is also readily seen that the first three rows are linearly independent. Thus the remaining eigenvalues λ_1, λ_2, and λ_3 are not zero. By the strict concavity of our optimization problem, we can have at most one negative eigenvalue. For, if not, we would have a continuum of initial values of the controls $c(t)$ and $u(t)$ that converge optimally to the same steady state, and therefore convex combinations of those paths will deliver higher utility. Also, one of the eigenvalues, say λ_3, must be negative, since otherwise the system will be unstable on the ratio K/h. From Santos (1990), we can conclude that the eigenvalues $\lambda_4 = 0$ and $\lambda_3 < 0$ correspond to the eigenvalues of the derivative of the policy function $Dg(K^*, h^*)$. Since g is homogeneous of degree one, these two eigenvalues are common to every steady state (K^*, h^*).

It follows from standard arguments (cf. Hirsch, Pugh, and Shub 1977, chap. 5) that at each steady state (K^*, h^*) there is a neighborhood containing a one-dimensional manifold crossing the ray of steady states at (K^*, h^*) such that, for all initial conditions (K_0, h_0) of this one-dimensional manifold, the optimal path converges exponentially to (K^*, h^*). Consequently, different initial conditions of the state variables $K(t)$ and $h(t)$ may lead asymptotically to different steady states. This means that two economies with different initial endowments of physical and human capital ($\tilde{K}(0) = K(0) = K_0$, $\tilde{h}(0) = h(0) = h_0$) will end up growing at the same rate, although the "levels" of $\tilde{K}(t)$ and $\tilde{h}(t)$ in these two economies, parameterized by the normalized variables $K(t)$ and $h(t)$, may remain different.

We can now summarize our results in this section in the following theorem.

THEOREM 1. Consider the dynamic optimization problem (P′) with the same assumptions as in proposition 1. Then:

a) Every positive initial condition (K_0, h_0) converges to some point on the ray of stationary states (K^*, h^*) determined by equations (14) and (16). Moreover, if $K_0/h_0 > K^*/h^*$, then the optimal orbit originating at (K_0, h_0) converges to a steady state (K^*, h^*) such that $K^* < K_0$. If $K_0/h_0 < K^*/h^*$, then the optimal orbit originating at (K_0, h_0) converges to a steady state (K^*, h^*) such that $K^* > K_0$.

b) At every steady state (K^*, h^*), the derivative of the policy function $Dg(K^*, h^*)$ has an eigenvalue $\lambda_4 = 0$ and another eigenvalue $\lambda_3 < 0$. Therefore, optimal paths approach the ray of steady states (K^*, h^*) at an exponential rate.

According to theorem 1, an increase in K will bring about a process of negative investment. In our simple economy, physical capital is readily transformed into consumption. A sudden increment in K lowers γ_1, the shadow price of the physical good. (The drop in γ_1 follows from the concavity of the value function and the fact that $V_K(K(0), h(0)) = \gamma_1(0)$.) As the shadow price equals the marginal utility (i.e., $\gamma_1(0) = c(0)^{-\sigma}$), a decrease in γ_1 results in a higher level of consumption. The economy then converges toward a new steady state, and along the optimal orbit the levels of both physical capital and consumption go down.[4] Hence, physical capital and consumption respond in a similar qualitative way as in the Cass-Koopmans model.

V. The Behavior of Human Capital Near Steady States

Although theorem 1 determines the dynamic evolution of physical capital, the evolution of the human capital variable is undetermined. Indeed, the following three cases are possible (fig. 3). *The normal case:* If the ratio of physical to human capital is higher (lower) than the steady-state solution, then the economy moves toward a steady state with a higher (lower) level of human capital. *The paradoxical case:* If the ratio of physical to human capital is higher (lower) than the steady-state solution, then the economy moves toward a steady state with a lower (higher) level of human capital. *The exogenous growth case:* Investment in human capital is insensitive to the ratio of physical to human capital.

Observe that an increase in physical capital from a stationary point m to a point a (see fig. 3) leads to the following transitional dynamics. (1) In the normal case, the economy moves toward a steady state with higher amounts of both physical and human capital than those of the steady-state solution m. (2) In the paradoxical case, the economy moves toward a steady state with lower amounts of both physical and human capital than those of the steady-state solution m. (3) In the exogenous growth case, the economy returns to the same steady state m. Although human capital here is an endogenous variable, it behaves in practice as an exogenous growth factor.

Conversely, an increase in human capital will give rise to the reverse transitional effects. Indeed, in the paradoxical case an increase in human capital will bring about a positive accumulation of both physical and human capital.

A sudden increase in K raises the relative price of human capital, $\gamma_2(0)/\gamma_1(0)$,[5] and the opportunity cost of labor, F_L. The increment in $\gamma_2(0)/\gamma_1(0)$ stimulates the time devoted to education, whereas the

[4] Consumption goes down since one can show from (18) that, independently of the direction of the eigenvectors, we must have $\dot{\gamma}_1(t) > 0$.

[5] This increase in $\gamma_2(0)/\gamma_1(0)$ can be shown by invoking the homogeneity of degree $1 - \sigma$ and strict concavity of V, and the fact that $DV(K(0), h(0)) = (\gamma_1(0), \gamma_2(0))$. Note that an increment in physical capital raises current and future productivity of human capital and lowers the real interest rate (as the rate of growth of consumption goes down). This combined effect makes more attractive the accumulation of human capital.

increment in F_L stimulates the time devoted to work. As a result, an increase in K has an undetermined effect on the accumulation of h.

Clearly, the change in F_L is determined in the margin by the elasticity of F_L with respect to K, and the change in $\gamma_2(0)/\gamma_1(0)$ should be linked to the elasticity σ. In the following theorem we prove that only these two parameter values determine the qualitative evolution of h near a steady state.

THEOREM 2. Consider the dynamic optimization problem (P′) with the same assumptions as in proposition 1. Let $\beta^* = F_{LK}(K^*, h^*)K^* \div F_L(K^*, h^*)$, where (K^*, h^*) is an interior steady state. Then (*a*) in the normal case, $\sigma \geq \beta^*$; (*b*) in the paradoxical case, $\sigma \leq \beta^*$; and (*c*) in the exogenous growth case, $\sigma = \beta^*$.

COROLLARY. (*a*) If $\sigma > \beta^*$, the economy belongs to the normal case; (*b*) if $\sigma < \beta^*$, the economy belongs to the paradoxical case.

A rather striking message of theorem 2 is that the qualitative behavior of h is independent of the discount rate, ρ, of the elasticity of the marginal productivity of the human capital technology, $\phi^* = G''(1 - u^*)(1 - u^*)/G'(1 - u^*)$, and of other relevant parameters of the model. The values of σ and β fully determine the sign of the evolution of h near a steady state. It follows that for a Cobb-Douglas technology with a constant depreciation rate, $AK^{\beta}L^{1-\beta} - \pi K$, $\pi \geq 0$, $0 < \beta < 1$, the normal case is always obtained for values $\sigma \geq 1$.

Proof of theorem 2. Given problem (P′), the first-order condition for the allocation of working time is given by

$$\gamma_1 F_L(K, N_0 uh) N_0 = \gamma_2 G'(1 - u). \tag{24}$$

By the implicit function theorem, it follows from (24) that

$$\frac{\partial u}{\partial K} = -\frac{\dfrac{\partial}{\partial K}\left[\dfrac{\gamma_1}{\gamma_2} F_L(K, N_0 uh) N_0\right]}{\dfrac{\partial}{\partial u}\left[\dfrac{\gamma_1}{\gamma_2} F_L(K, N_0 uh) - G'(1 - u)\right]}. \tag{25}$$

We first confine ourselves to part *a*. Observe that in this case $\partial u/\partial K \leq 0$. Given the concavity of F and G, we can see that the sign of $\partial u/\partial K$ is identical to that of the numerator in the right-hand side of (25). Hence, taking logs inside the brackets of this expression, we obtain

$$\epsilon_{\gamma_1, K} - \epsilon_{\gamma_2, K} + \beta \leq 0, \tag{26}$$

where, for $i = 1, 2$, $\epsilon_{\gamma_i, K} = (\partial \gamma_i/\partial K)(K/\gamma_i)$ and $\beta = F_{LK}(K, N_0 uh)K \div F_L(K, N_0 uh)$.

We next show that at a steady state (K^*, h^*) the elasticity $\epsilon_{\gamma_2,K} \leq 0$. For this purpose, observe that in the normal case $u(t) < u^*$. Consequently, equation (19) implies that $\dot{\gamma}_2(t) \geq 0$. Furthermore, an increase in K from a given steady state (K^*, h^*) makes the economy move to a higher steady state, in which every price γ_i $(i = 1, 2)$ is lower than in the original steady state (K^*, h^*). This is possible only if γ_2 drops with the initial increase in K. Therefore, $\epsilon_{\gamma_2,K} \leq 0$.

Given that $DV(K, h) = (\gamma_1, \gamma_2)$, the homogeneity of degree $-\sigma$ of the derivative of the value function entails that

$$V_{KK}(K, h)K + V_{Kh}(K, h)h = -\sigma\gamma_1. \tag{27}$$

As $V_{KK}(K, h) = \partial\gamma_1/\partial K$, from (27),

$$\epsilon_{\gamma_1,K} = -\sigma - \frac{V_{Kh}(K, h)h}{\gamma_1}. \tag{28}$$

Since $V_{Kh}(K, h) = V_{hK}(K, h) = \partial\gamma_2/\partial K \leq 0$, it follows from expressions (26) and (28) that, at a steady state (K^*, h^*), in the normal case, $\sigma \geq \beta^*$. This shows part *a*. The proofs of parts *b* and *c* proceed analogously. The theorem is thus established.

As illustrated in the preceding section, the consumption of the physical good goes up with an increase in K. We now show that the elasticity of this change is also dependent on the previous parameter values.

THEOREM 3. Let $\epsilon_{c,K}$ be the elasticity of c with respect to K at a stationary state (K^*, h^*). Then (*a*) in the normal case, $\epsilon_{c,K} \leq 1$; (*b*) in the paradoxical case, $\epsilon_{c,K} \geq 1$; and (*c*) in the exogenous growth case, $\epsilon_{c,K} = 1$.

Proof. (*a*) As shown in the proof of theorem 2, in the normal case $V_{Kh} \leq 0$. Hence, from (28), the elasticity of γ_1 with respect to K is $\epsilon_{\gamma_1,K} \geq -\sigma$. Since the shadow price of the physical good equals the marginal utility, $\gamma_1(0) = c(0)^{-\sigma}$, it must hold that, in the normal case, $\epsilon_{c,K} \leq 1$. An analogous argument validates parts *b* and *c*. The proof is complete.

We close this section with a numerical exercise of an economy with a Cobb-Douglas production function and a linear technology for human capital accumulation. This is an economy of the type studied in Lucas (1988). Let $U(c) = c^{1-\sigma}/(1 - \sigma)$, $F(K, N_0uh) = AK^{\beta}(N_0uh)^{1-\beta}$, and $G(1 - u) = \delta(1 - u)$, where $(1 - \sigma)\delta < \rho - n + (1 - \sigma)\theta < \delta$, $\delta > 0$. Observe that condition (12) becomes simply $\rho + \pi - \sigma\theta > 0$.

Writing out these parametric functional forms into the dynamical system formed by equations (20)–(23), we get the steady-state values for the following variables:

$$u^* = \frac{1}{\sigma}\left[\frac{\rho - n}{\delta} - (1 - \sigma)\left(1 - \frac{\theta}{\delta}\right)\right], \tag{29}$$

$$\nu = \frac{1}{\sigma}(\delta - \theta - \rho + n), \tag{30}$$

$$\frac{c^*}{K^*} = \frac{1}{N_0}\left[\frac{\rho + \nu\sigma + (1 - \beta)\pi}{\beta} - \nu + n\right], \tag{31}$$

$$\frac{c^*}{h^*} = \left(\frac{A\beta}{\rho + \sigma\nu + \pi}\right)^{1/(1-\beta)}\left[\frac{\rho + \sigma\nu + (1 - \beta)\pi}{\beta} - \nu - n\right]u^*, \tag{32}$$

$$\frac{K^*}{h^*} = \left(\frac{A\beta}{\rho + \sigma\nu + \pi}\right)^{1/(1-\beta)} N_0 u^*. \tag{33}$$

Moreover, in this case the coefficients matrix of the linearized system around a steady state takes the form

$$M(c^*, K^*, h^*, u^*) =$$

$$\begin{pmatrix} 0 & \frac{-c^*}{K^*}\left[\frac{(1-\beta)(\rho+\sigma\nu+\pi)}{\sigma}\right] & \frac{c^*}{h^*}\left[\frac{(1-\beta)(\rho+\sigma\nu+\pi)}{\beta}\right] & \frac{c^*}{u^*}\left[\frac{(1-\beta)(\rho+\sigma\nu+\pi)}{\sigma}\right] \\ -N_0 & \rho - n - (1-\sigma)\nu & \frac{K^*}{h^*}\left[\frac{(1-\beta)(\rho+\sigma\nu+\pi)}{\beta}\right] & \frac{K^*}{u^*}\left[\frac{(1-\beta)(\rho+\sigma\nu+\pi)}{\beta}\right] \\ 0 & 0 & 0 & -\delta h^* \\ -N_0\left(\frac{u^*}{K^*}\right) & N_0\left[\frac{u^* c^*}{(K^*)^2}\right] & 0 & \delta u^* \end{pmatrix}$$

As shown in the previous section, one of the eigenvalues of this matrix must be negative. Moreover, the eigenvector space belonging to this negative eigenvalue will determine the qualitative behavior of the variables c, K, h, and u around a stationary point.

We first consider the following benchmark economy: $N_0 = 1$, $A = 1$, $\rho = 0.05$, $n = 0.015$, $\delta = 0.05$, $\sigma = 1.5$, $\beta = 0.3$, $\pi = 0.01$, and $\theta = 0$. These parameters conform roughly to standard empirical evidence. Observe that the steady-state ratios c^*/h^* and K^*/h^* depend on the parameter A (see [32] and [33]). For simplicity the parameter A was set equal to one.

Letting $h^* = 1$, we obtain from (29)–(33) the steady-state values $c^* = 1.246$, $K^* = 5.797$, $u^* = 0.8$, and $\nu = 0.01$. Also, the negative eigenvalue of the matrix $M(c^*, K^*, h^*, u^*)$ is equal to -0.175, and $(-0.3116, -4.050, 0.1025, 0.3590)$ is an associated eigenvector.

As $\sigma > \beta$, the economy belongs to the normal case. This agrees with the signs of the second and third coordinates of the eigenvector that determine the evolution of K and h, respectively. However, one can see that the third coordinate is relatively close to zero. Hence, the transitional dynamics of this economy resemble the exogenous growth case. Indeed, from the eigenvector above, let $\dot{K} = -4.050$ and $\dot{h} = 0.1025$. Then computing the ratio $(\dot{h}/h^*)/(\dot{K}/K^*)$, we obtain that this magnitude is approximately equal to -0.147. This fairly inelastic response of h to increases in K may be explained in our model by the fact that $\dot{K}$ and c enter additively into the resource constraint (10) and that K does not enter into the technology for human capital accumulation (11). This latter case is the subject of the following section.

VI. Physical Capital in the Production of Education

In this section we consider a variant of the preceding model in which physical capital is also employed as an input of the production of human capital. That is, G is now an increasing function of both K and h. Our main finding is that if G is intensive in K, then both the exogenous growth and paradoxical cases can be ruled out. However, an example of a Cobb-Douglas technology shows that the three growth patterns may still arise in the opposite case in which G is intensive in h.

From the objective in (P′), we now consider, instead of (10) and (11), the resource constraints

$$\dot{K}(t) = F(v(t)K(t), N_0 u(t)h(t)) - (\nu + n + \pi)K(t) - N_0 c(t) \quad (34)$$

and

$$\dot{h}(t) = G\left(\frac{[1 - v(t)]K(t)}{N_0}, [1 - u(t)]h(t)\right) - (\nu + \theta)h(t), \quad (35)$$

where $v(t)$ is the fraction of physical capital devoted to the production of the consumption good. We suppose that both F and G are increasingly monotone in their arguments, concave, and linearly homogeneous. The first-order necessary conditions for an interior optimal solution are given by

$$c(t)^{-\sigma} = \gamma_1(t), \tag{36}$$

$$\gamma_1(t) F_K(v(t)K(t), N_0 u(t) h(t)) = \frac{\gamma_2(t)}{N_0} G_K\left(\frac{[1 - v(t)]K(t)}{N_0}, [1 - u(t)]h(t)\right), \tag{37}$$

$$\gamma_1(t) F_L(v(t)K(t), N_0 u(t) h(t)) = \frac{\gamma_2(t)}{N_0} G_L\left(\frac{[1 - v(t)]K(t)}{N_0}, [1 - u(t)]h(t)\right), \tag{38}$$

$$\frac{\dot{\gamma}_1(t)}{\gamma_1(t)} = \rho + \sigma\nu - F_K(v(t)K(t), N_0 u(t) h(t)) + \pi, \tag{39}$$

$$\frac{\dot{\gamma}_2(t)}{\gamma_2(t)} = \rho - n + \sigma\nu - G_L\left(\frac{[1 - v(t)]K(t)}{N_0}, [1 - u(t)]h(t)\right) + \theta, \tag{40}$$

where $F_K(\cdot, \cdot)$ and $G_K(\cdot, \cdot)$ denote the derivatives of the functions F and G with respect to the first argument.

Under conditions similar to those of Section III, one could again study the dynamics of off-balanced paths. We shall not pursue further the stability problem here. Our main concern, however, is to explore in this setting the existence of the exogenous growth and paradoxical cases near a steady state $\{c^*, K^*, h^*, v^*, u^*\}$. We shall make use of the following assumption.

ASSUMPTION A. The production function G is more intensive in physical capital than the production function F: If

$$\frac{F_K(v^*K^*, N_0 u^* h^*)}{F_L(v^*K^*, N_0 u^* h^*)} = \frac{G_K\left(\frac{(1 - v^*)K^*}{N_0}, (1 - u^*)h^*\right)}{G_L\left(\frac{(1 - v^*)K^*}{N_0}, (1 - u^*)h^*\right)},$$

then

$$\frac{v^*K^*}{N_0 u^* h^*} < \frac{(1 - v^*)K^*}{N_0(1 - u^*)h^*}.$$

Assume that an increase in K from a given steady-state solution m moves the economy to a point such as a of figure 3. Observe that after such an increase in K the price γ_1/γ_2 cannot go up. Moreover, the paradoxical and exogenous growth cases both require that at least either v or u go up. We shall now see, however, that both v and u must go down. The following basic effects can be singled out.

a) The factor intensity effect (the Rybczynski theorem).—An increase in K stimulates the production in the sector intensive in K and decreases production in the other sector. This result follows from the properties of the contract curve derived from equating the ratios of the marginal productivities of F and G (e.g., equalities [37] and [38]). Hence, assumption A implies that v and u will go down.

b) The price effect.—An increase in K may lower the relative price γ_1/γ_2. As a result, the production of human capital may go up and the production of the physical good may go down. Hence, the possible drop in γ_1/γ_2 makes both v and u go down.

These two effects combined together show us, therefore, that an increase in K brings about a process of human capital accumulation. Also, from the two effects and the fact that consumption should go up, an increase in K will set up a process of decumulation of physical capital. We may then conclude that the paradoxical and exogenous growth cases can be ruled out under the assumption that the educational sector is intensive in K (assumption A). If, however, the educational sector is intensive in human capital, both cases may still become true. This is illustrated in the Appendix with a Cobb-Douglas technology. For $F(v(t)K(t), N_0u(t)h(t)) = A[v(t)K(t)]^{\beta}[N_0u(t)h(t)]^{1-\beta}$ and

$$G\left(\frac{[1 - v(t)]K(t)}{N_0}, [1 - u(t)]h(t)\right)$$

$$= \delta\left\{\frac{[1 - v(t)]K(t)}{N_0}\right\}^{\alpha} \{[1 - u(t)]h(t)\}^{1-\alpha},$$

if $\alpha > \beta$, the function G is more intensive in physical capital. In this case, the paradoxical and exogenous growth cases are not possible. However, as $\alpha \rightarrow 0$, the dynamical system generated by this model converges in the C^1 functional sense to that of a Lucas-type model with $G(1 - u(t)) = \delta[1 - u(t)]$, as the one considered in Section V. Hence, a continuity argument shows that with free mobility of K and h across sectors, the paradoxical and exogenous growth cases are also plausible.

VII. Concluding Remarks

This paper has analyzed a generalized version of the Lucas-Uzawa model of endogenous growth with physical and human capital. Given a constant returns to scale technology for producing the consumption good, we have characterized a set of necessary and sufficient conditions on technologies and preferences that allow for the existence of balanced paths. Under these conditions, we have also established the global dynamics of optimal paths.

Our main result (theorem 1) states that economies with high ratios of physical to human capital will always decumulate physical capital, and economies with low ratios of physical to human capital will always increase their holdings of physical capital. This places human capital as a key factor for growth.

An injection of human capital from a given steady-state solution will always lead the economy to another steady state with higher levels of both physical capital and consumption. An injection of physical capital, however, will eventually bring about higher levels of both physical capital and consumption only if it induces a movement of labor from the production sector to the educational sector (the normal case). It is nevertheless possible to obtain the reverse effect (the paradoxical case), in which the economy converges to a balanced equilibrium with lower levels of both physical capital and consumption. If the allocation of labor between production and education is insensitive to the level of physical capital (the exogenous growth case), then under our normalization of the state variables the economy will converge back to the same steady state. In this case, the statistical properties regarding income growth and physical investment of the endogenous growth model are observationally equivalent to the standard one-sector model augmented with exogenous technological progress.

It remains an empirical issue to investigate which of these cases is most important for applied work. Our characterization in theorem 2 indicates that one should observe the normal case in which human capital has a positive response to an increase in the level of physical investment. The value attached to this response in our numerical computations was, however, relatively small. Hence, it may be very difficult to say in practice which of these three regions is statistically most relevant.

An empirical study by Mankiw, Romer, and Weil (1992) supports evidence in favor of the exogenous growth case. However, contrary to what the authors seem to suggest, we would like to emphasize that this is not necessarily a test of exogenous versus endogenous growth models. In our simple setting, the exogenous growth case is a particular instance of the endogenous growth framework.

If it is true that the accumulation of human capital is rather insensitive to variations in physical investment, a government will find that inflows of physical capital are eventually nullified by the actions of the consumption sector. In such a situation, the economy converges back to a neighboring steady state. A further research topic is to examine which types of policies are best suited to achieve paths of higher growth. Another line of research is to explore the dynamics in more micro-founded models, with different types of knowledge, human capital, and education (e.g., Rustichini and Schmitz 1991).

A simple variation of the Lucas-Uzawa model was considered in Section VI, in which physical capital entered into the production of education. We have shown that if the educational sector is intensive in physical capital, then a sudden increase in physical capital will always result in an accumulation of human capital.

Our model can be extended to allow for external effects in the accumulation of human capital as in Lucas (1988). In this case the production function may be written as $F(\tilde{K}(t),\ N(t)u(t)\tilde{h}(t),\ \tilde{h}^a(t))$, where $\tilde{h}^a(t)$ is the average level of human capital in the economy. The competitive solution will obviously differ from the planned solution because no agent will take into account the external effects when deciding how much to invest in his own human capital. Lucas shows that with external effects the set of steady states is a nonlinear, one-dimensional manifold for both competitive and efficient equilibria. However, by continuity, our stability result still holds for sufficiently low external effects.

Finally, we must recognize that other sources of endogenous growth are possible besides schooling. For instance, we can consider models of "learning by doing" (Arrow 1962) in which it is precisely the time devoted to production that increases the productivity of workers, models with linear technologies (Barro 1990) or asymptotically linear technologies (Jones and Manuelli 1990), or the already mentioned models with externalities (Romer 1986; Lucas 1988; Azariadis and Drazen 1990). The techniques developed in this paper may also be useful to analyze these types of models. It should be stressed, however, that our stability results may not hold in more general contexts. For example, Chamley (1993) has shown that a multiplicity of steady states may arise under an externality (or learning effect) on the time devoted to schooling. Also, Becker, Murphy, and Tamura (1990) illustrate the multiplicity of steady states and the discontinuity of the policy function when population is endogenously determined. Likewise, in later work we have shown that several steady states are possible in the case in which time can be devoted to leisure activities.

Appendix

In this Appendix we first derive from the maximum principle the law of motion of the variables $\{c, K, h, v, u\}$ for a Cobb-Douglas version of the model sketched in Section VI. We then show that for suitable limiting parameter values the dynamics of this model converges asymptotically to that of a Lucas-type model.

Let $F(vK, N_0uh) = A(vK)^{\beta}(N_0uh)^{1-\beta}$ and

$$G\left(\frac{(1-v)K}{N_0}, (1-u)h\right) = \delta\left[\frac{(1-v)K}{N_0}\right]^{\alpha}[(1-u)h]^{1-\alpha}.$$

Differentiating (36), we obtain

$$-\sigma\frac{\dot{c}(t)}{c(t)} = \frac{\dot{\gamma}_1(t)}{\gamma_1(t)}. \tag{A1}$$

Thus

$$\dot{c}(t) = \frac{c(t)}{\sigma}[\rho + \sigma\nu - F_K(v(t)K(t), N_0u(t)h(t)) + \pi], \tag{A2}$$

which is the same as equation (20). Moreover, making use of (40) and (A1), we obtain, differentiating (38),

$$\begin{aligned}&-\sigma\frac{\dot{c}(t)}{c(t)} + (\beta-\alpha)\frac{\dot{K}(t)}{K(t)} - (\beta-\alpha)\frac{\dot{h}(t)}{h(t)}\\ &+\left[\frac{\beta}{v(t)} + \frac{\alpha}{1-v(t)}\right]\dot{v}(t) - \left[\frac{\beta}{u(t)} + \frac{\alpha}{1-u(t)}\right]\dot{u}(t)\\ &= \rho - n + \sigma\nu - G_L\left(\frac{[1-v(t)]K(t)}{N_0}, [1-u(t)]h(t)\right) + \theta.\end{aligned} \tag{A3}$$

Observe that if we substitute out the values of $\dot{c}(t)$, $\dot{K}(t)$, and $\dot{h}(t)$, then (A3) becomes analogous to equation (23). Moreover, as $\alpha \rightarrow 0$, equations (34), (35), (A2), and (A3) converge in the C^1 functional sense to (20)–(23), with $G(1-u(t)) = \delta[1-u(t)]$, $G'(1-u(t)) = \delta$, $\beta(t) = \beta$, and $\phi(t) = 0$. This can be shown after some straightforward computations. For example, concerning equation (A3), we have in this Cobb-Douglas case

$$v(t) = \frac{\hat{\beta}u(t)}{\hat{\alpha} + (\hat{\beta} - \hat{\alpha})u(t)}, \tag{A4}$$

$$\delta\left\{\frac{[1-v(t)]K(t)}{N_0}\right\}^{\alpha}\{[1-u(t)]h(t)\}^{1-\alpha} \rightarrow \delta[1-u(t)]h(t) \quad \text{as } \alpha \rightarrow 0, \tag{A5}$$

$$(1-\alpha)\delta\left\{\frac{[1-v(t)]K(t)}{N_0}\right\}^{\alpha}\{[1-u(t)]h(t)\}^{-\alpha} \rightarrow \delta \quad \text{as } \alpha \rightarrow 0, \tag{A6}$$

where (A4) comes from (37) and (38), with $\hat{\beta} = \beta/(1-\beta)$ and $\hat{\alpha} = \alpha/(1-\alpha)$. Hence, $v(t) \rightarrow 1$, $dv(t)/du(t) \rightarrow 0$, and $\dot{v}(t) \rightarrow 0$ as $\alpha \rightarrow 0$. Moreover, after we substitute out the values of $\dot{c}(t)$, $\dot{K}(t)$, and $\dot{h}(t)$, in (A3) it is now a mechanical exercise to check that, as $\alpha \rightarrow 0$, equation (A3) converges to (23), with $F(K(t), N_0u(t)h(t)) = AK(t)^{\beta}[N_0u(t)h(t)]^{1-\beta}$ and $G(1-u(t)) = \delta[1-u(t)]$. Likewise, given that there is convergence on the ray of steady states, the linearization of (A3) converges to the linearization of (23). (These computations are available on request from the authors.)

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