

Constrained School Choice ^{*}

Guillaume Haeringer[†] Flip Klijn[‡]

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[†]Departament d'Economia i d'Història Econòmica, Universitat Autònoma de Barcelona, Spain; e-mail: guillaume.haeringer@uab.es

[‡]*Corresponding author.* Institute for Economic Analysis (CSIC), Campus UAB, 08193 Bellaterra (Barcelona), Spain; e-mail: flip.klijn@iae.csic.es

Abstract

Recently, several school districts in the US have adopted or consider adopting the Student-Optimal Stable mechanism or the Top Trading Cycles mechanism to assign children to public schools. There is evidence that for school districts that employ (variants of) the so-called Boston mechanism the transition would lead to efficiency gains. The first two mechanisms are strategy-proof, but in practice student assignment procedures typically impede a student to submit a preference list that contains all his acceptable schools. We study the preference revelation game where students can only declare up to a fixed number of schools to be acceptable. We focus on the stability and efficiency of the Nash equilibrium outcomes. Our main results identify rather stringent necessary and sufficient conditions on the priorities to guarantee stability or efficiency of either of the two mechanisms. This stands in sharp contrast with the Boston mechanism which has been abandoned in many US school districts but nevertheless yields stable Nash equilibrium outcomes.

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1 Introduction

School choice is referred in the literature on education as giving parents a say in the choice of the schools their children will attend. A recent paper by Abdulkadiroğlu and Sönmez (2003) has led to an upsurge of enthusiasm in the use of matching theory for the design and study of school choice mechanisms.¹ Abdulkadiroğlu and Sönmez (2003) discuss critical flaws of the current procedures of some school districts in the US to assign children to public schools, pointing out that the widely used Boston mechanism has the serious shortcoming that it is not in the parents' best interest to reveal their true preferences. Using a mechanism design approach, they propose and analyze two alternative student assignment mechanisms that do not have this shortcoming: the Student-Optimal Stable mechanism and the Top Trading Cycles mechanism.

A common practice in real-life school choice situations consists of asking to submit a preference list containing only a limited number of schools. For instance, in the school district of New York City each year more than 90,000 students are assigned to about 500 school programs, and parents are asked to submit a preference list containing at most 12 school programs. Until 2006 parents in Boston could not submit more than 5 schools in their choice list.² In Spain and in Hungary students applying to a college cannot submit a choice list containing more than 8 and 4 academic programs, respectively.³ This restriction is reason for concern. Imposing a curb on the length of the submitted lists compels participants to adopt a strategic behavior when choosing which ordered list to submit. For instance, if a participant fears rejection by his most preferred programs, it can be advantageous not to apply to these programs and use instead its allowed application slots for less preferred programs.

The matching literature usually assumes that individuals submit their true preferences when either the Student-Optimal Stable mechanism or the Top Trading Cycles mecha-

¹Recent papers include Abdulkadiroğlu (2005), Abdulkadiroğlu, Pathak, and Roth (2005, 2008), Abdulkadiroğlu, Pathak, Roth, and Sönmez (2005), Chen and Sönmez (2006), Erdil and Ergin (2008), Ergin and Sönmez (2006), Kesten (2005), and Pathak and Sönmez (2008).

²Abdulkadiroğlu, Pathak, and Roth (2005) report that in New York about 25% of the students submit a preference list containing the maximal number of school programs, which suggests that the constraint is binding for a significant number of students. Interestingly enough, the school district of Boston recently adopted the Student-Optimal Stable mechanism without a constraint on the length of submittable preference lists for the school year 2007–2008 (see Abdulkadiroğlu, Pathak, Roth and Sönmez (2006)).

³In Spain and Hungary colleges are not strategic, for the priority orders are determined by students' grades. So college admission in these countries is, strictly speaking, akin to school choice.

nism is used. However, little is known about the properties of these two mechanisms (i.e., the structure of equilibrium profiles and equilibrium outcomes) when individuals do not necessarily submit their true preferences. In this paper we aim at filling this gap by exploring the effects of imposing a quota (*i.e.*, a maximal length) on the submittable preference lists of students. Thereby we revive an issue that was initially raised by Romero-Medina (1998), who shows that any stable matching can be sustained at some Nash equilibrium under the Student-Optimal Stable mechanism.⁴ In this paper we consider and compare three matching mechanisms that are or have been employed or proposed in many US school districts: the Boston (BOS), the Student Optimal Stable Matching (SOSM) and the Top Trading Cycles (TTC) mechanisms.

The model considered in this paper is the school choice problem (Abdulkadiroğlu and Sönmez, 2003) where a number of students has to be assigned to a number of schools, each of which has a limited seat capacity. Students have preferences over schools and remaining unassigned and schools have exogenously given priority rankings over students.⁵ We introduce a preference revelation game where students can only declare up to a fixed number (the quota) of schools to be acceptable. Each possible quota, from 1 up to the total number of schools, together with a student assignment mechanism induces a strategic “quota-game.” Since the presence of the quota eliminates the existence of a dominant strategy when the mechanism at hand is the SOSM or TTC, we focus our analysis on the Nash equilibria of the quota-games. Regarding SOSM, our approach complements the work of Roth (1984), Gale and Sotomayor (1985a), and Alcalde (1996) who characterized the set of Nash equilibrium outcomes when the schools are strategic.⁶ As for TTC, so far little has been known about its Nash equilibria.

Our preliminary results concern the existence and the structure of the Nash equilibria under BOS, SOSM, and TTC. For all three mechanisms and for any quota, there are Nash equilibria in pure strategies. We establish that for the three mechanisms the associated

⁴Kojima and Pathak (2007) consider the game played by schools when for each student only a small set of schools is acceptable.

⁵Priorities are the counterpart of schools’ preferences over students in the college admissions problem (Gale and Shapley, 1962).

⁶Roth (1984) and Gale and Sotomayor (1985a) characterized the set of Nash equilibrium outcomes when schools are strategic agents in a college admissions problem, assuming that students truthfully reveal their (whole) preferences. In particular, they showed that Nash equilibria yield stable matchings and that any stable matching can be obtained as a Nash equilibrium outcome. Alcalde (1996) went one step further assuming that students may not necessarily use their weakly dominant strategy. He showed that the set of Nash equilibrium outcomes coincides with the set of individually rational matchings.

quota-games have a common feature: the equilibria are nested with respect to the quota. More precisely, given a quota, any Nash equilibrium is also a Nash equilibrium under any less stringent quota. This leads to the following important observation: If a Nash equilibrium outcome in a quota-game has an undesirable property then this is not simply due to the presence of a constraint on the size of submittable lists. Regarding BOS and TTC we obtain a much stronger result: Nash equilibrium outcomes are independent of the quota. This is a powerful result, since it allows us to reduce the analysis to games of quota 1.

The core of this paper is devoted to the analysis of equilibrium outcomes, focusing on stability and efficiency. Abdulkadiroğlu and Sönmez (2003) discuss in detail the importance and desirability of these two properties in school choice. In this paper, we explore under which conditions the mechanisms implement stable and efficient matchings in Nash equilibria. Most of our analysis will concentrate on SOSM and TTC. The results for BOS either are already known or come as byproducts of the characterizations for SOSM.⁷

Stability is the central concept in the two-sided matching literature and does not lose its importance in the closely related model of school choice.⁸ Loosely speaking, stability of an assignment obtains when, for any student, all the schools he prefers to the one he is assigned to have exhausted their capacity with students that have higher priority. Romero-Medina (1998) claims that any Nash equilibrium outcome under SOSM is stable. We provide an example that shows that this is not true.⁹ Furthermore, the unstable equilibrium outcome we present cannot be Pareto ranked with respect to the set of stable assignments, thereby leaving us with little hope for hitting on a closed form characterization of equilibrium outcomes under SOSM. We therefore turn to the problem of implementing stable matchings under SOSM. This turns out to be possible if, and only if, schools' priorities satisfy Ergin's (2002) acyclicity condition. However, we may understand this as a negative result, for Ergin's acyclicity is a condition that is likely not to be met in real-life school choice problems. As for BOS, it is easy to show that the correspondence of stable matchings is implemented in Nash equilibria. Finally, and for the sake of completeness, we also consider the stability of equilibrium outcomes under

⁷Alcalde (1996), Ergin and Sönmez (2006) and Pathak and Sönmez (2008) provide an extensive analysis of the equilibria under BOS.

⁸In many centralized labor markets, clearinghouses are most often successful if they produce stable matchings —see Roth (2002) and the references therein.

⁹See also Example 3 in Sotomayor (1998), which even applies to a larger class of mechanisms than SOSM.

TTC. Like for SOSM, under TTC unstable matchings may obtain in equilibrium. We show that Kesten’s (2006) acyclicity condition is necessary and sufficient to implement stable matchings under TTC.

In a school choice problem, efficiency is defined with respect to the preferences of students only. It is known that in this case SOSM may not be efficient —see Ergin (2002). TTC then becomes the most natural mechanism to obtain efficient matchings provided students do submit their true preferences. However, notice that since TTC is no longer strategy-proof when choice is constrained it is not clear whether it performs better than BOS, which is also efficient with respect to submitted preference lists. The efficiency of TTC turns out to be not robust to the Nash equilibrium operator. In fact, it is easy to see that an inefficient matching can be sustained by an Nash equilibrium, even if we restrict to undominated strategies. This negative result motivates then the search for conditions that ensure efficiency. We show that efficient Nash equilibrium outcomes can be guaranteed if, and only if, schools’ priorities satisfy a new acyclicity condition called X -acyclicity. This condition roughly states that two schools cannot prioritize differently two students that compete for the last available seat in both schools. A similar but slightly stronger condition, strong X -acyclicity, is necessary and sufficient to guarantee efficient Nash equilibrium outcomes under both SOSM and BOS. It may come as a surprise that we find the same necessary and condition for SOSM and BOS. However, since any stable matching can be obtained as a Nash equilibrium outcome under both SOSM and BOS, strong X -acyclicity needs to guarantee that there is a unique stable matching (otherwise the lattice structure of the set of stable matchings implies that not all equilibrium outcomes are efficient). In fact, nothing more is needed: X -acyclicity *is* a necessary and sufficient condition for the set of stable matchings to be a singleton.

The remainder of the paper is organized as follows. In Section 2, we recall the model of school choice. In Section 3, we describe the three mechanisms. In Section 4, we introduce the strategic game induced by the imposition of a quota on the revealed preferences. In Section 5 we provide existence results and establish the nestedness of equilibrium outcomes. In Sections 6 and 7 we investigate the implementability of stable and efficient matchings, respectively. In Section 8, we study Nash equilibria in undominated truncation strategies for the Student-Optimal Stable mechanism and the Top Trading Cycles mechanism. Finally, in Section 9, we discuss the policy implications of our results and our contribution to the literature on school choice. Almost all proofs are relegated to the Appendices.

2 School Choice

Following Abdulkadiroğlu and Sönmez (2003) we define a *school choice problem* by a set of schools and a set of students, each of which has to be assigned a seat at not more than one of the schools. Each student is assumed to have strict preferences over the schools and the option of remaining unassigned. Each school is endowed with a strict priority ordering over the students and a fixed capacity of seats. Formally, a *school choice problem* is a 5-tuple (I, S, q, P, f) that consists of

1. a set of *students* $I = \{i_1, \dots, i_n\}$,
2. a set of *schools* $S = \{s_1, \dots, s_m\}$,
3. a *capacity* vector $q = (q_{s_1}, \dots, q_{s_m})$,
4. a profile of strict *student preferences* $P = (P_{i_1}, \dots, P_{i_n})$, and
5. a strict *priority structure* of the schools over the students $f = (f_{s_1}, \dots, f_{s_m})$.

We denote by i and s a generic student and a generic school, respectively. An agent is an element of $V := I \cup S$. A generic agent is denoted by v . With a slight abuse of notation we write v for singletons $\{v\} \subseteq V$.

The preference relation P_i of student i is a linear order over $S \cup i$, where i denotes his outside option (*e.g.*, going to a private school). Student i prefers school s to school s' if $sP_i s'$. School s is *acceptable* to i if $sP_i i$. Henceforth, when describing a particular preference relation of a student we will only represent acceptable schools. For instance, $P_i = s, s'$ means that student i 's most preferred school is s , his second best s' , and any other school is unacceptable. For the sake of convenience, if all schools are unacceptable for i then we sometimes write $P_i = i$ instead of $P_i = \emptyset$. Let R_i denote the weak preference relation associated with the preference relation P_i .

The *priority ordering* f_s of school s assigns ranks to students according to their priority for school s . The *rank* of student i for school s is $f_s(i)$. Then, $f_s(i) < f_s(j)$ means that *student i has higher priority (or lower rank) for school s than student j* . For $s \in S$ and $i \in I$, we denote by $U_s^f(i)$ the set of students that have higher priority than student i for school s , *i.e.*, $U_s^f(i) = \{j \in I : f_s(j) < f_s(i)\}$.

Throughout the paper we fix the set of students I and the set of schools S . Hence, a *school choice problem* is given by a triple (P, f, q) , and simply by P when no confusion is possible.

School choice is closely related to the college admissions model (Gale and Shapley, 1962). The only but key difference between the two models is that in school choice

schools are mere “objects” to be consumed by students, whereas in the college admissions model (or more generally, in two-sided matching) both sides of the market are agents with preferences over the other side. In other words, a college admissions problem is given by 1–4 above and 5’. a profile of strict school preferences $P_S = (P_{s_1}, \dots, P_{s_m})$, where P_s denotes the strict preference relation of school $s \in S$ over the students.

Priority orderings in school choice can be reinterpreted as school preferences in the college admissions model. Therefore, many results or concepts for the college admissions model have their natural counterpart for school choice.¹⁰ In particular, an outcome of a school choice or college admissions problem is a *matching* $\mu : I \cup S \rightarrow 2^I \cup S$ such that for any $i \in I$ and any $s \in S$,

- $\mu(i) \in S \cup i$,
- $\mu(s) \in 2^I$,
- $\mu(i) = s$ if and only if $i \in \mu(s)$, and
- $|\mu(s)| \leq q_s$.

For $v \in V$, we call $\mu(v)$ agent v ’s allotment. For $i \in I$, if $\mu(i) = s \in S$ then student i is *assigned* a seat at school s under μ . If $\mu(i) = i$ then student i is *unassigned* under μ . For convenience we often write a matching as a collection of sets. For instance, $\mu = \{\{i_1, i_2, s_1\}, \{i_3\}, \{i_4, s_2\}\}$ denotes the matching in which students i_1 and i_2 each are assigned a seat at school s_1 , student i_3 is unassigned, and student i_4 is assigned a seat at school s_2 .

A key property of matchings in the two-sided matching literature is stability. Informally, a matching is stable if, for any student, all the schools he prefers to the one he is assigned to have exhausted their capacity with students that have higher priority. Formally, let P be a school choice problem. A matching μ is stable if

- it is *individually rational*, *i.e.*, for all $i \in I$, $\mu(i) R_i i$,
- it is *non wasteful* (Balinski and Sönmez, 1999), *i.e.*, for all $i \in I$ and all $s \in S$, $s P_i \mu(i)$ implies $|\mu(s)| = q_s$, and
- there is no *justified envy*, *i.e.*, for all $i, j \in I$ with $\mu(j) = s \in S$, $s P_i \mu(i)$ implies $f_s(j) < f_s(i)$.

We denote the set of individually rational matchings by $IR(P)$, the set of non wasteful matchings by $NW(P)$, and the set of stable matchings by $S(P)$.

¹⁰See, for instance, Balinski and Sönmez (1999).

Another desirable property for a matching is Pareto-efficiency. In the context of school choice, to determine whether a matching is Pareto-efficient we only take into account students' welfare. A matching μ' Pareto dominates a matching μ if all students prefer μ' to μ and there is at least one student that strictly prefers μ' to μ . Formally, μ' *Pareto dominates* μ if $\mu'(i)R_i\mu(i)$ for all $i \in I$, and $\mu'(i')P_{i'}\mu(i')$ for some $i' \in I$. A matching is *Pareto-efficient* if it is not Pareto dominated by any other matching. We denote the set of Pareto-efficient matchings by $PE(P)$.

A (student assignment) *mechanism* systematically selects a matching for each school choice problem. A mechanism is individually rational if it always selects an individually rational matching. Similarly, one can speak of non wasteful, stable, or Pareto-efficient mechanisms. Finally, a mechanism is *strategy-proof* if no student can ever benefit by unilaterally misrepresenting his preferences.¹¹

3 Three Competing Mechanisms

In this section we describe the mechanisms that we study in the context of constrained school choice: the Boston, Student-Optimal Stable, and the Top Trading Cycles mechanisms. The three mechanisms are direct mechanisms, *i.e.*, students only need to report an ordered list of their acceptable schools. For a profile of revealed preferences the matching that is selected by a mechanism is computed via an algorithm. Below we give a description of the three algorithms, thereby introducing some additional notation. Let (I, S, q, P, f) be a school choice problem.

3.1 The Boston Algorithm

The Boston mechanism was first described in the literature by Alcalde (1996) who called it the “Now-or-never” mechanism. The term “Boston mechanism” was coined by Abdulkadiroğlu and Sönmez (2003) because the mechanism was used in the Boston school district until recently. Consider a profile of ordered lists Q submitted by the students. The Boston algorithm finds a matching through the following steps.

STEP 1: Set $q_s^1 := q_s$ for all $s \in S$. Each student i proposes to the school that is ranked first in Q_i (if there is no such school then i remains unassigned). Each school s assigns

¹¹In game theoretic terms, a mechanism is strategy-proof if truthful preference revelation is a weakly dominant strategy.

up to q_s^1 seats to its proposers one at a time following the priority order f_s . Remaining students are rejected. Let q_s^2 denote the number of available seats at school s . If $q_s^2 = 0$ then school s is removed.

STEP l , $l \geq 2$: Each student i that is rejected in Step $l - 1$ proposes to the next school in the ordered list Q_i (if there is no such school then i remains unassigned). School s assigns up to q_s^l seats to its (new) proposers one at a time following the priority order f_s . Remaining students are rejected. Let q_s^{l+1} denote the number of available seats at school s . If $q_s^{l+1} = 0$ then school s is removed.

The algorithm stops when no student is rejected or all schools have been removed. Any remaining student remains unassigned. Let $\beta(Q)$ denote the matching. The mechanism β is the *Boston mechanism*, or *BOS* for short. It is well known that BOS is individually rational, non wasteful, and Pareto-efficient. It is, however, neither stable nor strategy-proof.

3.2 The Gale-Shapley Deferred Acceptance Algorithm

The deferred acceptance (DA) algorithm was introduced by Gale and Shapley (1962). Let Q be a profile of ordered lists submitted by the students. The DA algorithm finds a matching through the following steps.

STEP 1: Each student i proposes to the school that is ranked first in Q_i (if there is no such school then i remains unassigned). Each school s tentatively assigns up to q_s seats to its proposers one at a time following the priority order f_s . Remaining students are rejected.

STEP l , $l \geq 2$: Each student i that is rejected in Step $l - 1$ proposes to the next school in the ordered list Q_i (if there is no such school then i remains unassigned). Each school s considers the new proposers and the students that have a (tentative) seat at s . School s tentatively assigns up to q_s seats to these students one at a time following the priority order f_s . Remaining students are rejected.

The algorithm stops when no student is rejected. Each student is assigned to his final tentative school. Let $\gamma(Q)$ denote the matching. The mechanism γ is the *Student-Optimal Stable mechanism*, or *SOSM* for short. SOSM is a stable mechanism that is Pareto superior to any other stable matching mechanism (Gale and Shapley, 1962). An additional important property of SOSM is that it is strategy-proof (Dubins and Freedman, 1981; Roth, 1982b). However, it is not Pareto-efficient.

3.3 The Top Trading Cycles Algorithm

The Top Trading Cycles mechanism in the context of school choice was introduced by Abdulkadiroğlu and Sönmez (2003).¹² Let Q be a profile of ordered lists submitted by the students. The Top Trading Cycles algorithm finds a matching through the following steps.

STEP 1: Set $q_s^1 := q_s$ for all $s \in S$. Each student i points to the school that is ranked first in Q_i (if there is no such school then i points to himself, *i.e.*, he forms a *self-cycle*). Each school s points to the student that has the highest priority in f_s . There is at least one cycle. If a student is in a cycle he is assigned a seat at the school he points to (or to himself if he is in a self-cycle). Students that are assigned are removed. If a school s is in a cycle and $q_s^1 = 1$, then the school is removed. If a school s is in a cycle and $q_s^1 > 1$, then the school is not removed and its capacity becomes $q_s^2 := q_s^1 - 1$.

STEP l , $l \geq 2$: Each student i that is rejected in Step $l - 1$ points to the next school in the ordered list Q_i that has not been removed at some step r , $r < l$, or points to himself if there is no such school. Each school s points to the student with the highest priority in f_s among the students that have not been removed at a step r , $r < l$. There is at least one cycle. If a student is in a cycle he is assigned a seat at the school he points to (or to himself if he is in a self-cycle). Students that are assigned are removed. If a school s is in a cycle and $q_s^l = 1$, then the school is removed. If a school s is in a cycle and $q_s^l > 1$, then the school is not removed and its capacity becomes $q_s^{l+1} := q_s^l - 1$.

The algorithm stops when all students or all schools have been removed. Any remaining student is assigned to himself. Let $\tau(Q)$ denote the matching. The mechanism τ is the *Top Trading Cycles mechanism*, or *TTC* for short. TTC is Pareto-efficient and strategy-proof (see Roth, 1982a, for a proof in the context of housing markets and Abdulkadiroğlu and Sönmez, 2003, for a proof in the context of school choice). The mechanism is also individually rational and non wasteful. However, it is not stable.

¹²The Top Trading Cycles mechanism was inspired by Gale's Top Trading Cycles algorithm which was used by Roth and Postlewaite (1977) to obtain the unique core allocation for housing markets (Shapley and Scarf, 1974). A variant of the Top Trading Cycles mechanism was introduced by Abdulkadiroğlu and Sönmez (1999) for a model of house allocation with existing tenants.

3.4 An Illustrative Example

We illustrate the impact of the quota on the length of submittable preference lists through the following example.

Let $I = \{i_1, i_2, i_3, i_4\}$ be the set of students, $S = \{s_1, s_2, s_3\}$ be the set of schools, and $q = (1, 2, 1)$ be the capacity vector. The students' preferences P and the priority structure f are given in the table below. So, for instance, $P_{i_1} = s_2, s_1, s_3$ and $f_{s_1}(i_1) < f_{s_1}(i_2) < f_{s_1}(i_3) < f_{s_1}(i_4)$.

P_{i_1}	P_{i_2}	P_{i_3}	P_{i_4}	f_{s_1}	f_{s_2}	f_{s_3}
s_2	s_1	s_1	s_2	i_1	i_3	i_4
s_1	s_2	s_2	s_3	i_2	i_4	i_1
s_3	s_3	s_3	s_1	i_3	i_1	i_2
				i_4	i_2	i_3

One easily verifies that if there is no quota on the length of submittable preference lists and if the students truthfully report their preference lists, then the mechanisms yield the following three matchings:

$$\begin{aligned} \beta(P) &= \{\{s_1, i_2\}, \{s_2, i_1, i_4\}, \{s_3, i_3\}\} \\ \gamma(P) &= \{\{s_1, i_1\}, \{s_2, i_3, i_4\}, \{s_3, i_2\}\} \\ \tau(P) &= \{\{s_1, i_3\}, \{s_2, i_1, i_4\}, \{s_3, i_2\}\}. \end{aligned}$$

Note that if in a direct revelation game under γ or τ students could only submit a list of 2 schools, student i_2 would remain unassigned (and the other students unaffected), provided that each student submits the truncated list with his two most preferred schools. Therefore, if students can only submit short preference lists, then (at least) student i_2 ought to strategize (*i.e.*, list school s_3) to ensure a seat at some (acceptable) school. In particular, the profile of truncated preferences does not constitute a Nash equilibrium. Under both mechanisms in the constrained setting, truncating one's true preferences is in general not a (weakly) dominant strategy.

4 Constrained Preference Revelation

Fix the priority ordering f and the capacities q . We consider the following school choice procedure. Students are asked to submit (simultaneously) preference lists $Q =$

$(Q_{i_1}, \dots, Q_{i_n})$ of “length” at most k (*i.e.*, preference lists with at most k acceptable schools). Here, k is a positive integer, $1 \leq k \leq m$, and is called the *quota*. Subsequently, a mechanism φ is used to obtain the matching $\varphi(Q)$ and for all $i \in I$, student i is assigned a seat at school $\varphi(Q)(i)$.

Clearly, the above procedure induces a strategic form game, the *quota-game* $\Gamma^\varphi(P, k) := \langle I, \mathcal{Q}(k)^I, P \rangle$. The set of players is the set of students I . The strategy set of each student is the set of preference lists with at most k acceptable schools and is denoted by $\mathcal{Q}(k)$. Let $\mathcal{Q} := \mathcal{Q}(m)$. Outcomes of the game are evaluated through the true preferences $P = (P_{i_1}, \dots, P_{i_n})$, where with some abuse of notation P denotes the straightforward extension of the preference relation over schools (and the option of remaining unassigned) to matchings. That is, for all $i \in I$ and matchings μ and μ' , $\mu P_i \mu'$ if and only if $\mu(i) P_i \mu'(i)$.

For any profile of preferences $Q \in \mathcal{Q}^I$ and any $i \in I$, we write Q_{-i} for the profile of preferences that is obtained from Q after leaving out preferences Q_i of student i . A profile of submitted preference lists $Q \in \mathcal{Q}(k)^I$ is a Nash equilibrium of the game $\Gamma^\varphi(P, k)$ (or *k-Nash equilibrium* for short) if for all $i \in I$ and all $Q'_i \in \mathcal{Q}(k)$, $\varphi(Q_i, Q_{-i}) R_i \varphi(Q'_i, Q_{-i})$. Let $\mathcal{E}^\varphi(P, k)$ denote the set of k -Nash equilibria. Let $\mathcal{O}^\varphi(P, k)$ denote the set of k -Nash equilibrium outcomes, *i.e.*, $\mathcal{O}^\varphi(P, k) := \{\varphi(Q) : Q \in \mathcal{E}^\varphi(P, k)\}$.

Remark 4.1 Setting the same quota for all students is without loss of generality since in the proofs we never compare the values of the quota for different students.

If the quota is smaller than the total number of schools, *i.e.*, $k < m$, then students typically cannot submit their true preference lists and hence there is no weakly dominant strategy for SOSM and TTC. The next result shows that nevertheless there is a class of undominated strategies.

One piece of advice about which preference list a student should submit follows from the strategy-proofness of the Student-Optimal Stable mechanism and the Top Trading Cycles mechanism in the unconstrained setting: it does not pay off to submit a list of schools that does not respect the true order. More precisely, a list that does not respect the order of a student’s true preferences is weakly dominated by listing the same schools in the “true order.” Let φ be a mechanism. Student i ’s strategy $Q_i \in \mathcal{Q}(k)$ in the game $\Gamma^\varphi(P, k)$ is *weakly k-dominated* by another strategy $Q'_i \in \mathcal{Q}(k)$ if $\varphi(Q'_i, Q_{-i}) R_i \varphi(Q_i, Q_{-i})$ for all $Q_{-i} \in \mathcal{Q}(k)^{I \setminus i}$.

Lemma 4.2 *Let P be a school choice problem. Let $1 \leq k \leq m$. Let $i \in I$ be a student. Consider two strategies $Q_i, Q'_i \in \mathcal{Q}(k)$ such that (a) Q_i and Q'_i contain the same set of*

schools, and (b) for any two schools s and s' listed in Q_i (or Q'_i), $sQ'_i s'$ implies $sP_i s'$. Then, Q_i is weakly k -dominated by Q'_i in the games $\Gamma^\gamma(P, k)$ and $\Gamma^\tau(P, k)$.

Proof Let $\varphi := \gamma, \tau$. The result follows directly from the strategy-proofness of γ (Dubins and Freedman, 1981; Roth, 1982b) and τ (Abdulkadiroğlu and Sönmez, 2003) by using Q'_i as student i 's “true preferences:” $\varphi(Q'_i, Q_{-i})(i)$ is ranked higher than $\varphi(Q_i, Q_{-i})(i)$ by Q'_i , hence $\varphi(Q'_i, Q_{-i})(i)$ is ranked higher than $\varphi(Q_i, Q_{-i})(i)$ by P_i . ■

The message of Lemma 4.2 is clear: a student cannot lose (and may possibly gain) by submitting the same set of schools in the true order.

5 Existence and Nestedness of Equilibria

Our main interest in this section is to analyze the extent to which Nash equilibria are affected by the value of the quota. To avoid vacuously true statements, we first establish the existence of (pure) Nash equilibria in any constrained school choice problem for all three mechanisms, for any value of the quota.¹³

Proposition 5.1 *For any school choice problem P and quota k , $\mathcal{E}^\varphi(P, k) \neq \emptyset$, for $\varphi = \beta, \gamma, \tau$.*

Understanding whether the presence of a quota affects the set of equilibria and equilibrium outcomes is crucial in our analysis of constrained school choice games. The next results describe how the equilibria vary when the quota changes. Fortuitously, Proposition 5.1 is a direct corollary to these results.

For BOS it turns out that the equilibrium outcomes do not depend on the quota.

Proposition 5.2 *For any school choice problem P and quota k , $\mathcal{O}^\beta(P, k) = \mathcal{O}^\beta(P, 1)$.*

The existence of equilibria under BOS is therefore a straightforward implication of Proposition 5.2 and $\mathcal{O}^\beta(P, m) \neq \emptyset$ (implied by Ergin and Sönmez, 2006, Theorem 1 or a slight adaptation of Alcalde, 1996, Theorem 4.6), where m is the number of schools. We do

¹³An adaptation of the arguments in the proofs of Alcalde (1996, Theorem 4.6) or Ergin and Sönmez (2006, Theorem 1) establishes Theorem 5.1 for the case of β (any k) and γ (with $k = 1$). Also notice that when $k = m$, the result for γ and τ follows from the strategy-proofness of the unconstrained mechanisms.

not give a proof of Proposition 5.2 as it is a direct consequence of Theorem 6.1, which provides a characterization of the equilibrium outcomes under BOS.

As for SOSM, an equilibrium (outcome) for a given value of the quota is also an equilibrium (outcome) for any higher value of the quota, *i.e.*, equilibria are nested.

Theorem 5.3 *For any school choice problem P and quotas $k < k'$, $\mathcal{E}^\gamma(P, k) \subseteq \mathcal{E}^\gamma(P, k')$.*

Notice that when $k = 1$ there is only one round in the Boston and the DA algorithm, and this round is the same for both algorithms. So, the existence of equilibria for SOSM for any value of the quota follows directly from the existence of equilibria for BOS.

Finally, the set of equilibrium outcomes under TTC is invariant with respect to the quota.

Theorem 5.4 *For any school choice problem P and quota k , $\mathcal{O}^\tau(P, k) = \mathcal{O}^\tau(P, 1)$.*

Notice that the true strategy profile P is an equilibrium for $k = m$ (because TTC is strategy-proof). Hence, Theorem 5.4 implies the existence of Nash equilibria for any value of the quota under TTC.

The fact that under BOS and TTC the set of equilibrium outcomes does not depend on the value of the quota will simplify to a great extent our analysis. Indeed, for these two mechanisms it will be enough to consider strategy profiles in which students submit a list containing at most one school. As for SOSM, the implications of Theorem 5.3 are not as sharp.

6 Implementation of Stable Matchings

We address in the section the question of the stability of equilibrium outcomes. Among the three mechanisms we consider, only SOSM is designed to produce stable matchings — provided agents are truthful. However, when agents are constrained it is not clear whether a particular mechanism, including SOSM, yields stable matching in equilibrium. While SOSM is the most natural candidate when studying the stability of equilibrium outcomes we also consider BOS and TTC.

We first start with BOS. Quite surprisingly, it turns out that any equilibrium outcome is stable.

Proposition 6.1 *For any school choice problem P and any quota k , the game $\Gamma^\beta(P, k)$ implements $S(P)$ in Nash equilibria, i.e., $\mathcal{O}^\beta(P, k) = S(P)$.*

This result is obtained through a straightforward adaptation of the proof of Theorem 1 in Ergin and Sönmez (2006). Alcalde (1996, Theorem 4.6) obtained a similar result in the context of a marriage market (*i.e.*, when both sides of the market are strategic agents) but without any constraint on the size of the submittable preference lists. A slight adaptation of his arguments also leads to a proof of Theorem 6.1. Its proof is therefore omitted.

We now turn to the analysis of equilibrium outcomes when the mechanism in use is SOSM. Since for quota $k = 1$ BOS and SOSM coincide, the games $\Gamma^\gamma(P, 1)$ and $\Gamma^\beta(P, 1)$ also coincide. Hence, Proposition 6.1 implies that the game $\Gamma^\gamma(P, 1)$ implements $S(P)$ in Nash equilibria, i.e., $\mathcal{O}^\gamma(P, 1) = S(P)$. Together with the nestedness of the equilibria under SOSM (Theorem 5.3) it follows that any stable matching can be obtained as an equilibrium outcome under SOSM, for any value of the quota.

Proposition 6.2 (Romero-Medina, 1998, Theorem 7) *For any school choice problem P and any quota k , $S(P) \subseteq \mathcal{O}^\gamma(P, k)$.*

However, the next example shows that for higher values of the quota, not all Nash equilibrium outcomes are necessarily stable.¹⁴

Example 6.3 An Unstable Nash Equilibrium Outcome in $\Gamma^\gamma(P, k)$

Let $I = \{i_1, i_2, i_3\}$ be the set of students, $S = \{s_1, s_2, s_3\}$ be the set of schools, and $q = (1, 1, 1)$ be the capacity vector. The students' preferences P and the priority structure f are given in the table below. Let $k = 2$ be the quota and $Q \in \mathcal{Q}(2)^I$ as given below.

P_{i_1}	P_{i_2}	P_{i_3}	f_{s_1}	f_{s_2}	f_{s_3}			
s₁	s₃	s₃	i_3	i_3	i_1	Q_{i_1}	Q_{i_2}	Q_{i_3}
s_3	s_1	s₂	i_1	i_1	i_2	s_1	s_1	s_3
s_2	s₂	s_1	i_2	i_2	i_3	s_3	s_2	s_1

One easily verifies that at $\gamma(Q) = \{\{i_1, s_1\}, \{i_2, s_2\}, \{i_3, s_3\}\}$ (which is indicated by the square boxes) student i_2 has justified envy for school s_3 . So, $\gamma(Q)$ is not stable. (In fact the unique stable matching is $\gamma(P) = \{\{i_1, s_1\}, \{i_2, s_3\}, \{i_3, s_2\}\}$, indicated in boldface.)

¹⁴We are not the first to provide an example with an unstable equilibrium outcome. Example 3 in Sotomayor (1998) already made this point for a class of mechanisms that includes SOSM. However, the generality of her example comes at the cost of using dominated strategies.

Routine computations show that i_2 has no profitable deviations. So, $Q \in \mathcal{E}^\gamma(P, 2)$. Notice also that $\gamma(Q)$ does not Pareto dominate $\gamma(P)$, nor is it Pareto dominated by $\gamma(P)$. Finally, and in view of Proposition 4.2, notice that (1) none of the students' strategies in the equilibrium exhibits "dominated reversals" of schools and (2) all students submit a preference list with the maximum number k of schools. \diamond

Example 6.3 and Theorem 5.3 suggest that unstable equilibrium outcomes are difficult to avoid in the quota-game under SOSM. Hence, the only degree of freedom that is left to obtain stable equilibrium outcomes is the schools' priority structure. The next result provides a condition on the priority structure under which SOSM implements the correspondence of stable matchings in Nash equilibria. The relevant condition is an acyclicity condition introduced by Ergin (2002). Loosely speaking, Ergin-acyclicity guarantees that no student can block a potential improvement for any other two students without affecting his own assignment.

Definition 6.4 Ergin-Acyclicity (Ergin, 2002)

Given a priority structure f , an *Ergin-cycle* is constituted of distinct $s, s' \in S$ and $i, j, l \in I$ such that the following two conditions are satisfied:

Ergin-cycle condition: $f_s(i) < f_s(j) < f_s(l)$ and $f_{s'}(l) < f_{s'}(i)$ and

ec-scarcity condition: there exist (possibly empty and) disjoint sets $I_s, I_{s'} \subseteq I \setminus \{i, j, l\}$ such that $I_s \subseteq U_s^f(j)$, $I_{s'} \subseteq U_{s'}^f(i)$, $|I_s| = q_s - 1$, and $|I_{s'}| = q_{s'} - 1$.

A priority structure is *Ergin-acyclic* if no Ergin-cycles exist. \triangle

Theorem 6.5 *Let $k \neq 1$. Then, f is an Ergin-acyclic priority structure if and only if for any school choice problem P , the game $\Gamma^\gamma(P, k)$ implements $S(P)$ in Nash equilibria, i.e., $\mathcal{O}^\gamma(P, k) = S(P)$.*

Ergin (2002) showed that Ergin-acyclicity of the priority structure is necessary and sufficient for the Pareto-efficiency of SOSM.¹⁵ Therefore, Theorem 6.5 shows that Ergin-acyclicity has a different impact depending on whether one considers SOSM *per se* or in the context of the induced preference revelation game.

¹⁵Ergin (2002) also showed that Ergin-acyclicity is sufficient for group strategy-proofness and consistency of SOSM as well as necessary for each of these conditions separately. In the setting of a two-sided matching model where also schools are strategic agents, Kesten (2007) showed that schools cannot manipulate by under-reporting capacities or by pre-arranged matches under SOSM if and only if the priority structure is Ergin-acyclic.

Obviously, TTC was not introduced to produce stable matchings. It is easy to construct an example for which not every equilibrium outcome is stable.

Example 6.6 A School Choice Problem P with $S(P) \cap \mathcal{O}^\tau(P, 1) = \emptyset$

Let $I = \{i_1, i_2, i_3\}$ be the set of students, $S = \{s_1, s_2\}$ be the set of schools, and $q = (1, 1)$ be the capacity vector. The students' preferences P and the priority structure f are given in the table below.

P_{i_1}	P_{i_2}	P_{i_3}	f_{s_1}	f_{s_2}
s_2	s_1	s_1	i_1	i_3
			i_2	i_2
			i_3	i_1

It is easy to check that the unique stable matching is $\mu = \{\{i_1, s_2\}, \{i_2, s_1\}, \{i_3\}\}$. We show that μ cannot be sustained at any Nash equilibrium of the game $\Gamma^\tau(P, 1)$. Suppose to the contrary that μ can be sustained at some Nash equilibrium. In other words, there is a profile $Q \in \mathcal{Q}(1)^I$ such that $\tau(Q) = \mu$ and $Q \in \mathcal{E}^\tau(P, 1)$. Since $\tau(Q) = \mu$, $Q_{i_1} = s_2$ and $Q_{i_2} = s_1$. If $Q_{i_3} = s_1$, then $\tau(Q)(i_3) = s_1 \neq \mu(i_3)$. So, $Q_{i_3} \neq s_1$. But then $\tau(Q')P_{i_3}\tau(Q)$ for $Q' := (Q_{i_1}, Q_{i_2}, s_1)$. Hence, $Q \notin \mathcal{E}^\tau(P, 1)$, a contradiction. \diamond

However, if we are to compare the three mechanisms we need to find a sufficient and necessary condition on the priority structure that guarantees stability, in very much the same way as we have done for SOSM. In the case of TTC the crucial necessary and sufficient condition for the stability of equilibrium outcomes is Kesten-acyclicity (2006).

Definition 6.7 Kesten-Acyclicity (Kesten, 2006)

Given a priority structure f , a *Kesten-cycle* is constituted of distinct $s, s' \in S$ and $i, j, l \in I$ such that the following two conditions are satisfied:

Kesten-cycle condition $f_s(i) < f_s(j) < f_s(l)$ and $f_{s'}(l) < f_{s'}(i), f_{s'}(j)$ and

kc-scarcity condition there exists a (possibly empty) set $I_s \subseteq I \setminus \{i, j, l\}$ with $I_s \subseteq U_s^f(i) \cup [U_s^f(j) \setminus U_s^f(l)]$ and $|I_s| = q_s - 1$.

A priority structure is *Kesten-acyclic* if no Kesten-cycles exist. \triangle

Kesten (2006) showed that Kesten-acyclicity of the priority structure is necessary and sufficient for the stability of the Top Trading Cycles mechanism when students report their true preferences.¹⁶ Kesten-acyclicity implies Ergin-acyclicity (Lemma 1, Kesten, 2006). It is easy to check that the reverse holds if all schools have capacity 1.

¹⁶Kesten (2006) also showed that Kesten-acyclicity is necessary and sufficient for the Top Trading

Theorem 6.8 *Let $1 \leq k \leq m$. Then, f is a Kesten-acyclic priority structure if and only if for any school choice problem P , the game $\Gamma^\tau(P, k)$ implements $S(P)$ in Nash equilibria, i.e., $\mathcal{O}^\tau(P, k) = S(P)$.*

Kesten’s (2006) result and Theorem 6.8 have in common that Kesten-acyclicity is both necessary and sufficient for the stability of the Top Trading Cycle mechanism. Yet, it is important to note that, contrary to Kesten (2006), in our game students typically cannot reveal their true preferences.

Theorem 6.8 is easily proved. If f is Kesten-cyclic then for some school choice problem P , $\tau(P)$ is not stable (Kesten, 2006, Theorem 1). Yet P is an m -Nash equilibrium (τ is strategy-proof), so by Theorem 5.4 for any k there is a k -Nash equilibrium Q with $\tau(Q) = \tau(P) \notin S(P)$. Conversely, if f is Kesten-acyclic then it is also Ergin-acyclic (Kesten, 2006, Lemma 1) and γ and τ coincide (Kesten, 2006, Theorem 1), so the stability of equilibrium outcomes under TTC follows from Theorem 6.5.

Since Kesten-acyclicity implies Ergin-acyclicity we can compare in a straightforward way the three mechanisms regarding the stability of equilibrium outcomes. If our criterion is determined by the domain of “problem-free” priority structures, then BOS outperforms SOSM, and SOSM in turn outperforms TTC.

7 Implementation of Pareto-Efficient Matchings

In this section we address the implementation of Pareto-efficient matchings. The main candidate in this case is TTC since it was designed to produce Pareto-efficient matchings —provided agents are truthful. To obtain a full comparison as in the previous section we shall also consider the other two mechanisms.

If we do not impose any restriction on the priority structure then equilibrium outcomes under TTC may not be Pareto-efficient. To see this, consider the following situation with 2 students and 2 schools, each with 1 seat. Let $P(i_1) = s_1, s_2$, $P(i_2) = s_2, s_1$, $f_{s_1}(i_2) < f_{s_1}(i_1)$, and $f_{s_2}(i_1) < f_{s_2}(i_2)$. Then, $Q = (s_2, s_1)$ is a Nash equilibrium, but $\tau(Q) = \{\{i_1, s_2\}, \{i_2, s_1\}\}$ is not Pareto-efficient.¹⁷ The key element in this example is that student i_1 has higher priority at one school and student i_2 has higher priority at

Cycles mechanism to be resource monotonic and population monotonic. In addition, he also proved that the Top Trading Cycles mechanism coincides with the Student-Optimal Stable mechanism if and only if the priority structure is Kesten-acyclic.

¹⁷Note that neither $Q_{i_1} = s_2$ nor $Q_{i_2} = s_1$ are dominated strategies.

the other school. If one of the schools, say school s_1 , had a capacity greater than 1, say $c + 1$, then we would still be able to sustain a Pareto-inefficient equilibrium provided we can find a set of students not including i_2 that (1) can fill c seats at school s_1 and (2) have a higher priority at school s_1 than student i_2 . The condition we introduce below, X -acyclicity,¹⁸ formalizes this intuition.

Definition 7.1 X -Cycles and X -Acyclicity

Given a priority structure f , an X -cycle is constituted of distinct $s, s' \in S$ and $i, i' \in I$ such that the following two conditions are satisfied:

X-cycle condition: $f_s(i) < f_s(i')$ and $f_{s'}(i') < f_{s'}(i)$ and

xc-scarcity condition: there exist (possibly empty and) disjoint sets $I_s \subseteq I \setminus i$, $I_{s'} \subseteq I \setminus i'$ such that $I_s \subseteq U_s^f(i)$, $I_{s'} \subseteq U_{s'}^f(i')$, $|I_s| = q_s - 1$, and $|I_{s'}| = q_{s'} - 1$.

A priority structure is X -acyclic if no X -cycles exist. △

The next result establishes that X -acyclicity is a necessary and sufficient condition to guarantee that all equilibrium outcomes under TTC are Pareto-efficient.

Theorem 7.2 *Let $1 \leq k \leq m$. Then, f is an X -acyclic priority structure if and only if for any school choice problem P , all Nash equilibria of the game $\Gamma^\tau(P, k)$ are Pareto-efficient, i.e., $\mathcal{O}^\tau(P, k) \subseteq PE(P)$.*

We now consider the question of the efficiency of equilibrium outcomes under BOS and SOSM. First of all, note that under either mechanism any stable matching can be sustained at some equilibrium (see the discussion after Proposition 6.1). Hence, by the lattice structure of the set of stable matchings, BOS and SOSM typically induce Pareto-inefficient equilibrium outcomes. We now provide a necessary and sufficient condition that ensures that all equilibrium outcomes under BOS and SOSM are Pareto-efficient. Note that although the set of equilibrium outcomes under BOS is a subset of the set of equilibrium outcomes under SOSM, the condition is the same for both mechanisms. This does not come as a big surprise since for both mechanisms we have to make sure that the set of stable matchings is a singleton (otherwise there is a Pareto-inefficient equilibrium outcome).

The relevant condition is a slight variant of X -acyclicity.

¹⁸The X represents the cross in the priority structure that results from connecting the two entries of i_1 and the two entries of i_2 .

Definition 7.3 Weak X -Cycles and Strong X -Acyclicity

Given a priority structure f , a weak X -cycle is constituted of distinct $s, s' \in S$ and $i, i' \in I$ such that the following two conditions are satisfied:

X-cycle condition: $f_s(i) < f_s(i')$ and $f_{s'}(i') < f_{s'}(i)$ and

wxc-scarcity condition: there exist (possibly empty and) disjoint sets $I_s \subseteq I \setminus i, I'_s \subseteq I \setminus i'$ such that $I_s \subseteq U_s^f(i')$, $I'_s \subseteq U_{s'}^f(i)$, $|I_s| = q_s - 1$, and $|I'_s| = q_{s'} - 1$.

A priority structure is strongly X -acyclic if no weak X -cycles exist. △

Remark 7.4 Note that if $((s, s'), (i, j, l))$ constitutes an Ergin-cycle, then $((s, s'), (i, l))$ constitutes a weak X -cycle. Hence, strong X -acyclicity implies Ergin-acyclicity.

Clearly, strong X -acyclicity is very restrictive. In fact, it is easy to see that strong X -acyclicity implies both X -acyclicity and Ergin-acyclicity. Nevertheless it is a necessary (and sufficient) condition to guarantee the Pareto-efficiency of all equilibrium outcomes under SOSM as well as BOS.

Theorem 7.5 *Let f be a priority structure. Let $1 \leq k \leq m$. Then, the following are equivalent:*

- (i) f is strongly X -acyclic.
- (ii) For any school choice problem P , $S(P)$ is a singleton.
- (iii) For any school choice problem P , all Nash equilibria of the game $\Gamma^\gamma(P, k)$ are Pareto-efficient, i.e., $\mathcal{O}^\gamma(P, k) \subseteq PE(P)$.
- (iv) For any school choice problem P , all Nash equilibria of the game $\Gamma^\beta(P, k)$ are Pareto-efficient, i.e., $\mathcal{O}^\beta(P, k) \subseteq PE(P)$.

Since strong X -acyclicity implies X -acyclicity we can also compare the three mechanisms regarding the Pareto-efficiency of equilibrium outcomes. If our criterion is determined by the domain of “problem-free” priority structures, then TTC outperforms both SOSM and BOS, and SOSM performs equally well as BOS.

Remark 7.6 Below we show that apart from [Kesten-acyclicity \Rightarrow Ergin-acyclicity], [strong X -acyclicity \Rightarrow Ergin-acyclicity], and [strong X -acyclicity \Rightarrow X -acyclicity], there are no other logical implications regarding pairs of acyclicity conditions. The Venn diagram in Figure 1 summarizes these facts.¹⁹ Each node indicates the existence of a priority structure that satisfies the associated requirements.

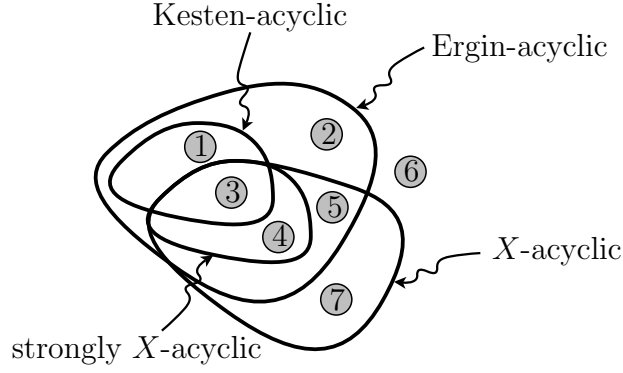


Figure 1: Venn diagram of the acyclicity conditions

We have made use of four different acyclicity conditions. Below we indicate the absence or existence of each of the 12 possible logical implications between pairs of acyclicity conditions.

- [Kesten-acyclicity \Rightarrow Ergin-acyclicity]: Kesten (2006, Lemma 1);
- \neg [Kesten-acyclicity \Rightarrow Ergin-acyclicity]: Example 2 in Kesten (2006) which is given by $f_{s_1} = i_1, i_2, i_3$ and $f_{s_2} = i_3, i_1, i_2$, $q_{s_1} = 1$ and $q_{s_2} = 2$;
- [strong X -acyclicity \Rightarrow X -acyclicity]: immediate from definition;
- \neg [X -acyclicity \Rightarrow strong X -acyclicity]: Example 2 in Kesten (2006) which is given by $f_{s_1} = i_1, i_2, i_3$ and $f_{s_2} = i_3, i_1, i_2$, $q_{s_1} = 1$ and $q_{s_2} = 2$;
- [strong X -acyclicity \Rightarrow Ergin-acyclicity]:
if $((s, s'), (i, j, l))$ constitutes an Ergin-cycle, then $((s, s'), (i, l))$ constitutes a weak X -cycle;
- \neg [strong X -acyclicity \Rightarrow Kesten-acyclicity]:
 $f_{s_1} = i_1, i_2, i_3$ and $f_{s_2} = i_3, i_1, i_2$, $q_{s_1} = 1$ and $q_{s_2} = 3$;
- \neg [X -acyclicity \Rightarrow Kesten-acyclicity]: follows from [strong X -acyclicity \Rightarrow X -acyclicity] and \neg [strong X -acyclicity \Rightarrow Kesten-acyclicity];

¹⁹A proof that [Kesten-acyclicity and X -acyclicity \Rightarrow strong X -acyclicity] is as follows. Suppose that the priority structure is Kesten-acyclic, X -acyclic, but not strongly X -acyclic. By Theorem 7.5(ii) \Rightarrow (i), there is a school choice problem P with $|S(P)| \geq 2$. By Theorem 6.8, $S(P) = \mathcal{O}^\tau(P, k)$. By Theorem 7.2, $\mathcal{O}^\tau(P, k) \subseteq PE(P)$. Hence, $S(P) \subseteq PE(P)$ and $|S(P)| \geq 2$, which contradicts the optimality of the Student-Optimal Stable matching.

- \neg [Ergin-acyclicity \Rightarrow X -acyclicity]:
 $f_{s_1} = i_1, i_2$ and $f_{s_2} = i_2, i_1, q_{s_1} = q_{s_2} = 1$;
- \neg [Ergin-acyclicity \Rightarrow strong X -acyclicity]: follows from [strong X -acyclicity \Rightarrow X -acyclicity] and \neg [Ergin-acyclicity \Rightarrow X -acyclicity];
- \neg [Kesten-acyclicity \Rightarrow X -acyclicity]:
 $f_{s_1} = i_1, i_2$ and $f_{s_2} = i_2, i_1, q_{s_1} = q_{s_2} = 1$;
- \neg [Kesten-acyclicity \Rightarrow strong X -acyclicity]: follows from [strong X -acyclicity \Rightarrow X -acyclicity] and \neg [Kesten-acyclicity \Rightarrow X -acyclicity];
- \neg [X -acyclicity \Rightarrow Ergin-acyclicity]:
 $f_{s_1} = i_1, i_4, i_5, i_2, i_3$ and $f_{s_2} = i_2, i_4, i_5, i_3, i_1, q_{s_1} = q_{s_2} = 2$.

Finally, we give an example of a priority structure that satisfies the requirements that are associated with each of the 7 nodes in Figure 1. Note that all examples can be extended to incorporate additional students or schools by (1) giving additional students lower priority and (2) introducing multiple copies of the priority ordering of an existing school.

1. $f_{s_1} = i_1, i_2$ and $f_{s_2} = i_2, i_1, q_{s_1} = q_{s_2} = 1$;
2. $f_{s_1} = i_1, i_2, i_3, i_4$ and $f_{s_2} = i_3, i_1, i_4, i_2, q_{s_1} = 1$ and $q_{s_2} = 2$;
3. any f where schools have identical priority over students;
4. $f_{s_1} = i_1, i_2, i_3$ and $f_{s_2} = i_3, i_1, i_2, q_{s_1} = 1$ and $q_{s_2} = 3$;
5. Example 2 in Kesten (2006) which is given by $f_{s_1} = i_1, i_2, i_3$ and $f_{s_2} = i_3, i_1, i_2, q_{s_1} = 1$ and $q_{s_2} = 2$;
6. $f_{s_1} = i_1, i_2, i_3$ and $f_{s_2} = i_3, i_2, i_1, q_{s_1} = 1$ and $q_{s_2} = 1$;
7. $f_{s_1} = i_1, i_4, i_5, i_2, i_3$ and $f_{s_2} = i_2, i_4, i_5, i_3, i_1, q_{s_1} = q_{s_2} = 2$.

8 Equilibria in Truncations

In this section we focus on “truncation” strategies which are shown to be undominated in the quota-games induced by both the Student-Optimal Stable mechanism and the Top Trading Cycles mechanism. We first strengthen the negative side of Theorems 6.5 and 6.8 by providing an example that admits a strong Nash equilibrium in truncations that induces an unstable matching. Next, again for both mechanisms, we will show that in general there is also no relation between the set of unassigned students at equilibrium and the set of unassigned students in stable matchings. However, for Nash equilibria in truncations we do obtain a positive result in this respect for the Student-Optimal Stable mechanism.

A *truncation* of a preference list P_i is a list P'_i obtained from P_i by deleting some school and all less preferred acceptable schools.²⁰ The following lemma says that in the games $\Gamma^\gamma(P, k)$ and $\Gamma^\tau(P, k)$ submitting a truncation “as long as possible” is k -undominated. Formally, student i 's strategy $Q_i \in \mathcal{Q}(k)$ is k -dominated by another strategy $Q'_i \in \mathcal{Q}(k)$ if $\varphi(Q'_i, Q_{-i}) R_i \varphi(Q_i, Q_{-i})$ for all $Q_{-i} \in \mathcal{Q}(k)^{I \setminus i}$ and $\varphi(Q'_i, Q'_{-i}) P_i \varphi(Q_i, Q'_{-i})$ for some $Q'_{-i} \in \mathcal{Q}(k)^{I \setminus i}$. A strategy in $\mathcal{Q}(k)$ is k -undominated if it is not k -dominated by any other strategy in $\mathcal{Q}(k)$.

Lemma 8.1 *Let P be a school choice problem. Let $1 \leq k \leq m$. Let $i \in I$ be a student. Denote the number of (acceptable) schools in P_i by $|P_i|$. Then, the strategy P_i^k of submitting the first $\min\{k, |P_i|\}$ schools of the true preference list P_i in the true order is k -undominated in the games $\Gamma^\gamma(P, k)$ and $\Gamma^\tau(P, k)$.*

Although the strategy profile $P^k := (P_i^k)_{i \in I}$ is a profile of k -undominated strategies, it is not necessarily a Nash equilibrium in the games $\Gamma^\gamma(P, k)$ and $\Gamma^\tau(P, k)$. In case it is a Nash equilibrium it may still induce an unstable matching as Example 8.2 shows.

Example 8.2 For both γ and τ : A Strong Nash Equilibrium in (Undominated) Truncations that yields an Unstable Matching

Let $I = \{i_1, i_2, i_3, i_4\}$ be the set of students, $S = \{s_1, s_2, s_3\}$ be the set of schools, and $q = (1, 1, 1)$ be the capacity vector. The students' preferences P and the priority structure f are given in the table below. Let $k = 2$ be the quota and $Q \in \mathcal{Q}(2)^I$ as given below.

²⁰Truncations have been studied by Roth and Vande Vate (1991), Roth and Rothblum (1999), and Ehlers (2004) and have also appeared in practice (see for instance Mongell and Roth, 1991).

P_{i_1}	P_{i_2}	P_{i_3}	P_{i_4}	f_{s_1}	f_{s_2}	f_{s_3}
s_1	s_2	s_3	s_1	i_3	i_1	i_2
s_2	s_3	s_1	s_2	i_1	i_2	i_4
s_3	s_1	s_2	s_3	i_2	i_3	i_3
				i_4	i_4	i_1

Q_{i_1}	Q_{i_2}	Q_{i_3}	Q_{i_4}
s_1	s_2	s_3	s_1
s_2	s_3	s_1	s_2

One easily verifies that $\gamma(Q) = \{\{i_1, s_1\}, \{i_2, s_2\}, \{i_3, s_3\}, \{i_4\}\}$. Since student i_4 has justified envy for school s_3 , $\gamma(Q) \notin S(P)$. It remains to show that $Q \in \mathcal{E}^\gamma(P, 2)$. Since students i_1 , i_2 , and i_3 are assigned a seat at their favorite school, it is sufficient to check that student i_4 has no profitable deviation. Notice that the only possibility for student i_4 to change the outcome of the mechanism is by listing school s_3 . So, the only strategies that we have to check are given by $\bar{Q}(2) = \{Q^a, Q^b, Q^c, Q^d, Q^e\}$, where $Q^a = s_3$, $Q^b = s_1, s_3$, $Q^c = s_2, s_3$, $Q^d = s_3, s_1$, and $Q^e = s_3, s_2$. Routine computations show that none of these strategies is a profitable deviation. So, $Q \in \mathcal{E}^\gamma(P, 2)$.²¹

Furthermore, since students i_1 , i_2 , and i_3 are assigned a seat at their favorite school at $\gamma(Q)$ and $Q \in \mathcal{E}^\gamma(P, 2)$, it follows that Q is a strong Nash equilibrium (*cf.* Aumann, 1959) in $\Gamma^\gamma(P, 2)$.

As for the Top Trading Cycles mechanism, one easily verifies that also $\tau(Q) = \{\{i_1, s_1\}, \{i_2, s_2\}, \{i_3, s_3\}, \{i_4\}\}$. For the same reason as before, it is sufficient to check that student i_4 has no profitable deviation. This, however, is immediate since student i_4 cannot “break” the cycle $(i_1, s_1, i_3, s_3, i_2, s_2)$ that forms in the first step of the TTC algorithm. Hence, Q is also a strong Nash equilibrium in $\Gamma^\tau(P, 2)$. \diamond

The results of McVitie and Wilson (1970) and Roth (1984b) for college admissions imply that for any school choice problem the set of unassigned students is the same for all stable matchings.²² In other words, for $\mu, \mu' \in S(P)$, $\mu(i) = i$ implies $\mu'(i) = i$. Given the restrictiveness of the acyclicity conditions to guarantee stable Nash equilibrium outcomes, one may wonder whether at least always the set of unassigned students at equilibrium coincides with the set of unassigned students in stable matchings. In fact, a less ambitious

²¹Note that it is not necessary to set the quota equal to 2. Strategy profile Q is also a Nash equilibrium in the unconstrained setting, *i.e.*, when the quota is $k = 3$. Finally, one can straightforwardly extend the example for $m > 3$ and/or $n > 4$ by making existing schools unacceptable for new students and new schools unacceptable for existing students.

²²A generalization of this result is known in the two-sided matching literature as the “Rural Hospital Theorem” (Roth, 1986) and says that the degree of occupation and quality of interns at typically less demanded rural hospitals in the US is not due to the choice of a specific stable matching.

idea would be to establish that at equilibrium the *number* of unassigned students equals the number of unassigned students in stable matchings. The following two examples show that in general this is not true. In other words, the number of unassigned students at equilibrium is not inherited from that of the set of stable matchings. Given Proposition 6.2, this in particular implies for the Student-Optimal Stable mechanism that the number of unassigned students can vary from one equilibrium outcome to another.

Example 8.3 For both γ and τ : Less Assigned Students in an Equilibrium than in Stable Matchings

Let $I = \{i_1, i_2, i_3\}$ be the set of students, $S = \{s_1, s_2, s_3\}$ be the set of schools, and $q = (1, 1, 1)$ be the capacity vector. The students' preferences P and the priority structure f are given in the table below. One easily verifies that strategy profile Q given below is a Nash equilibrium in $\Gamma^\gamma(P, 2)$ and $\Gamma^\tau(P, 2)$.

P_{i_1}	P_{i_2}	P_{i_3}	f_{s_1}	f_{s_2}	f_{s_3}			
s_1	s_3	s_3	i_3	i_2	i_1			
s_3	s_1	s_2	i_1	i_3	i_2			
s_2		s_1	i_2	i_1	i_3			

Q_{i_1}	Q_{i_2}	Q_{i_3}
s_1	s_1	s_3
s_3		s_1

Since $\gamma(Q) = \tau(Q) = \{\{i_1, s_1\}, \{i_3, s_3\}, \{i_2\}, \{s_2\}\}$ and $\gamma(P) = \{\{i_1, s_1\}, \{i_2, s_3\}, \{i_3, s_2\}\}$, there are less assigned students at $\gamma(Q) = \tau(Q)$ than in any stable matching. \diamond

Example 8.4 For both γ and τ : More Assigned Students in an Equilibrium than in Stable Matchings

Let $I = \{i_1, i_2, i_3\}$ be the set of students, $S = \{s_1, s_2, s_3\}$ be the set of schools, and $q = (1, 1, 1)$ be the capacity vector. The students' preferences P and the priority structure f are given in the table below. One easily verifies that strategy profile Q given below is a Nash equilibrium in $\Gamma^\gamma(P, 2)$ and $\Gamma^\tau(P, 2)$.

P_{i_1}	P_{i_2}	P_{i_3}	f_{s_1}	f_{s_2}	f_{s_3}			
s_2	s_3	s_3	i_3	i_2	i_1			
	s_2	s_2	i_1	i_3	i_2			
	s_1	s_1	i_2	i_1	i_3			

Q_{i_1}	Q_{i_2}	Q_{i_3}
s_2	s_3	s_1
s_3	s_2	s_2

Since $\gamma(Q) = \tau(Q) = \{\{i_1, s_2\}, \{i_2, s_3\}, \{i_3, s_1\}\}$ and $\gamma(P) = \{\{i_2, s_3\}, \{i_3, s_2\}, \{i_1\}, \{s_1\}\}$, there are more assigned students at $\gamma(Q) = \tau(Q)$ than in any stable matching. \diamond

We do obtain a positive result for γ if we restrict ourselves to equilibria in truncations. More precisely, the following proposition says that if a profile of truncations is a Nash equilibrium in the game $\Gamma^\gamma(P, k)$ then the set of assigned students at the equilibrium coincides with the set of assigned students at any stable matching. In fact, each Nash equilibrium in truncations in the game $\Gamma^\gamma(P, k)$ yields a matching that is either the student-optimal stable matching $\gamma(P)$ or Pareto dominates $\gamma(P)$. For a matching μ , denote $M(\mu)$ for the set of assigned students, *i.e.*, $M(\mu) := \{i \in I : \mu(i) \neq i\}$.

Proposition 8.5 *Let P be a school choice problem. Let $1 \leq k \leq m$. If $P^k \in \mathcal{E}^\gamma(P, k)$, then $M(\gamma(P^k)) = M(\gamma(P))$. In fact, $\gamma(P^k)R_i\gamma(P)$ for all $i \in I$.*

For τ we cannot obtain a similar result as the following proposition shows.

Proposition 8.6 *Let P be a school choice problem. Let $1 \leq k \leq m$. If $P^k \in \mathcal{E}^\tau(P, k)$, then possibly $|M(\tau(P^k))| < |M(\gamma(P))|$ or $|M(\tau(P^k))| > |M(\gamma(P))|$.*

9 Discussion

We studied in this paper the stability and efficiency of Nash equilibrium outcomes in a school choice problem when either BOS, SOSM, or TTC is used. At first sight, the most robust mechanism is BOS, for Nash equilibrium outcomes are always stable. In all other cases we need to impose a condition on the priority structure to guarantee stability or efficiency.²³ The problem is that these conditions are very restrictive, and hence not likely to be met in practice.²⁴ Also, it is interesting to note that for SOSM, the implementability of efficient matchings implies the implementability of stable matchings (see Figure 1). This is not the case for TTC.

Presumably then, constraining students' choices is a very costly policy. It *de facto* forces them to strategize, which in turns may slash the designer or the policy maker's interest for using either SOSM or TTC. The results we obtained should be contrasted with experimental real-life data, however. From the experimental side, Calsamiglia, Haeringer, and Klijn (2008) show that constraining choices, although having a clear impact on the

²³Other recent papers on implementation in various settings of two-sided matching include Pais (2008), Shinotsuka and Takamiya (2003), Sotomayor (2003), and Suh (2003).

²⁴Another negative feature of SOSM and TTC is that there are equilibria that match (unmatch) students that are unassigned (assigned) at the stable matchings. In particular, the number of matched students may vary within the set of equilibrium outcomes — see the examples in Haeringer and Klijn (2008).

performance of the mechanisms, does not alter too much the relative hierarchy of SOSM, TTC, and BOS (in this order) when one is concerned with diverse issues such as stability, efficiency, truth-telling, or even social mobility. In 2003 the New York City Department of Education (NYCDOE) adopted a centralized mechanism based on SOSM (Abdulkadiroğlu, Pathak, and Roth, 2005). Although choice in this mechanism is constrained, Abdulkadiroğlu, Pathak, and Roth (2008) provide evidence that over the years participants learned how to make sound choices. Also, the school district of Boston removed the constraint on the length of submittable preference lists for the school year 2007–2008 (see Abdulkadiroğlu, Pathak, Roth and Sönmez (2006)). This suggests that a (reasonable) constraint would be desirable when the authorities change their mechanisms and adopt either SOSM or TTC, and then after a few years the constraint could be dropped.

From a theoretical perspective, one possible extension of our model is the incorporation of incomplete information. Ehlers and Massó (2008) study a many-to-one matching market with incomplete information. They show that at least for stable mechanisms (*i.e.*, in particular SOSM) there is a strong link between the ordinal Bayesian Nash equilibria under incomplete information and the Nash equilibria under complete information.²⁵ More precisely, Ehlers and Massó’s results show that a characterization of the equilibria under complete information immediately leads to a characterization of the equilibria under incomplete information.

A Appendix: Proofs for SOSM

Let $Q \in \mathcal{Q}^I$. We denote $DA(Q)$ for the application of the DA algorithm (with students proposing) to Q . We will make use of the following two results to prove Theorem 5.3.

Lemma A.1 (Roth, 1982b, Lemma 1; *cf.* Roth and Sotomayor 1990, Lemma 4.8)

Let $Q \in \mathcal{Q}^I$ and $i \in I$. Let $Q'_i \in \mathcal{Q}$ be a preference list whose first choice is $\gamma(Q)(i)$ if $\gamma(Q)(i) \neq i$, and the empty list otherwise. Then, $\gamma(Q'_i, Q_{-i})(i) = \gamma(Q)(i)$.

Lemma A.2 *For any school choice problem P and quota k , $\mathcal{O}^\gamma(P, k) \subseteq IR(P) \cap NW(P)$.*

Proof Let $Q \in \mathcal{E}^\gamma(P, k)$. It is immediate that $\gamma(Q) \in IR(P)$. Suppose $\gamma(Q) \notin NW(P)$. Then, there are $i \in I$ and $s \in S$ with $sP_i\gamma(Q)(i)$ and $|\gamma(Q)(s)| < q_s$. Let \bar{Q}_i be

²⁵A strategy profile is an ordinal Bayesian Nash equilibrium if it is a Bayesian Nash equilibrium for every von Neumann-Morgenstern utility representation of individuals’ true preferences.

the empty list. Let $\bar{Q} := (\bar{Q}_i, Q_{-i})$. By a result of Gale and Sotomayor (1985b, Theorem 2) extended to the college admissions model (Roth and Sotomayor, 1990, Theorem 5.34), for each $j \in I \setminus i$, either $\gamma(\bar{Q})(j) = \gamma(Q)(j)$ or $\gamma(\bar{Q})(j)Q_j\gamma(Q)(j)$. Hence, the set of schools to which each $j \in I \setminus i$ proposes in $DA(\bar{Q})$ is a subset of the schools to which he proposes in $DA(Q)$. Since moreover \bar{Q}_i is the empty list, each school receives in $DA(\bar{Q})$ only a subset of the proposals of $DA(Q)$. For school s this immediately implies that $|\gamma(\bar{Q})(s)| \leq |\gamma(Q)(s)| < q_s$. So, if we take $Q'_i = s$ then $\gamma(Q'_i, Q_{-i})(i) = s$. Since $sP_i\gamma(Q)(i)$, Q'_i is a profitable deviation for i at Q in $\Gamma^\gamma(P, k)$. So, $Q \notin \mathcal{E}^\gamma(P, k)$, a contradiction. Hence, $\gamma(Q) \in NW(P)$. ■

Proof of Theorem 5.3 It suffices to prove the theorem for $k' = k+1$. Let $Q \in \mathcal{E}^\gamma(P, k)$ and suppose that $Q \notin \mathcal{E}^\gamma(P, k+1)$. Then, there is a student i and a strategy $Q'_i \in \mathcal{Q}(k+1)$ with $\gamma(Q'_i, Q_{-i})P_i\gamma(Q_i, Q_{-i})$. By Lemma A.2, $\gamma(Q) \in IR(P)$. Hence, $\gamma(Q'_i, Q_{-i})(i) \in S$. Note also that Q'_i must be a list containing exactly $k+1$ schools, for otherwise it would also be a profitable deviation in $\Gamma^\gamma(P, k)$, contradicting $Q \in \mathcal{E}^\gamma(P, k)$.

Let s be the last school listed in Q'_i . We claim that $\gamma(Q'_i, Q_{-i})(i) = s$. Suppose $\gamma(Q'_i, Q_{-i})(i) \neq s$. Consider the truncation of Q'_i after $\gamma(Q'_i, Q_{-i})(i)$ and denote this list by Q''_i . In other words, Q''_i is the list obtained from Q'_i by making all schools listed after $\gamma(Q'_i, Q_{-i})(i)$ unacceptable. By assumption, $Q''_i \in \mathcal{Q}(k)$. It follows from the DA algorithm that $\gamma(Q''_i, Q_{-i}) = \gamma(Q'_i, Q_{-i})$. Hence, Q''_i is a profitable deviation for i at Q in $\Gamma^\gamma(P, k)$, a contradiction. So, $\gamma(Q'_i, Q_{-i})(i) = s$.

Let $\hat{Q}_i := s$. Note $\hat{Q}_i \in \mathcal{Q}(k)$. By Lemma A.1, $\gamma(\hat{Q}_i, Q_{-i})(i) = s$. Hence, \hat{Q}_i is a profitable deviation for i at Q in $\Gamma^\gamma(P, k)$, a contradiction. Hence, $Q \in \mathcal{E}^\gamma(P, k+1)$. ■

We need the following three lemmas to prove Theorem 6.5.

Lemma A.3 *Let f be an Ergin-cyclic priority structure. Let $2 \leq k \leq m$. Then, there is a school choice problem P with an unstable equilibrium outcome in the game $\Gamma^\gamma(P, k)$, i.e., for some $Q \in \mathcal{E}^\gamma(P, k)$, $\gamma(Q) \notin S(P)$.*

Proof Since f is Ergin-cyclic, we may assume, without loss of generality, that

- (a) $f_{s_1}(i_1) < f_{s_1}(i_2) < f_{s_1}(i_3)$ and $f_{s_2}(i_3) < f_{s_2}(i_1)$,
- (b) $f_{s_1}(i_j) < f_{s_1}(i_2)$ for each $j \in I_1 := \{4, \dots, q_{s_1} + 2\}$, and
- (c) $f_{s_2}(i_j) < f_{s_2}(i_1)$ for each $j \in I_2 := \{q_{s_1} + 3, \dots, q_{s_1} + q_{s_2} + 1\}$.

Consider students' preferences P defined by $P_{i_1} := s_2, s_1$, $P_{i_2} := s_1$, $P_{i_3} := s_1, s_2$, $P_{i_j} := s_1$ for $j \in I_1$, $P_{i_j} := s_2$ for $j \in I_2$, and $P_{i_j} := \emptyset$ for all $j \in \{q_{s_1} + q_{s_2} + 2, \dots, n\}$.

We distinguish among three cases for the priority ordering f_{s_2} of school s_2 with respect to students i_1 , i_2 , and i_3 : (i) $f_{s_2}(i_2) < f_{s_2}(i_3) < f_{s_2}(i_1)$, (ii) $f_{s_2}(i_3) < f_{s_2}(i_2) < f_{s_2}(i_1)$, and (iii) $f_{s_2}(i_3) < f_{s_2}(i_1) < f_{s_2}(i_2)$. One easily verifies that in each of the cases (i), (ii), and (iii), the unique stable matching for P is $\mu^* = \gamma(P)$ with $\mu^*(i_1) = s_1$, $\mu^*(i_3) = s_2$, and $\mu^*(i_2) = i_2$.

Consider $Q \in \mathcal{Q}(k)^I$ defined by $Q_{i_2} := \emptyset$ and $Q_i := P_i$ for all $i \in I \setminus i_2$. One easily verifies that in each of the cases (i), (ii), and (iii), $\gamma(Q)(i_1) = s_2$ and $\gamma(Q)(i_3) = s_1$. So, $\gamma(Q) \neq \mu^*$, and hence $\gamma(Q) \notin S(P)$. Finally, one easily verifies that $Q \in \mathcal{E}^\gamma(P, k)$. ■

A mechanism is non bossy if no student can maintain his allotment and cause a change in the other students' allotments by reporting different preferences.

Definition A.4 Non Bossy Mechanism (Satterthwaite and Sonnenschein, 1981)

A mechanism φ is *non bossy* if for all $i \in I$, $Q_i, Q'_i \in \mathcal{Q}$, and $Q_{-i} \in \mathcal{Q}^{I \setminus i}$, $\varphi(Q'_i, Q_{-i})(i) = \varphi(Q_i, Q_{-i})(i)$ implies $\varphi(Q'_i, Q_{-i}) = \varphi(Q_i, Q_{-i})$. \triangle

Lemma A.5 *Let f be an Ergin-acyclic priority structure. Then, γ is non bossy.*

Proof Follows from Ergin's (2002) Theorem 1, (iv) \Rightarrow (iii) and proof of (iii) \Rightarrow (ii). ■

Lemma A.6 *Let f be an Ergin-acyclic priority structure. Let $2 \leq k \leq m$. Then, for any school choice problem P all equilibrium outcomes in the game $\Gamma^\gamma(P, k)$ are stable, i.e., for all $Q \in \mathcal{E}^\gamma(P, k)$, $\gamma(Q) \in S(P)$.*

Proof Suppose to the contrary that $Q \in \mathcal{E}^\gamma(P, k)$ but $\gamma(Q) \notin S(P)$. So, by Lemma A.2, there are $i, j \in I$, $i \neq j$ and $s \in S$ with $\gamma(Q)(j) = s$, $sP_i\gamma(Q)(i)$, and $f_s(i) < f_s(j)$.

Since γ is strategy-proof in the unconstrained setting (i.e., when the quota equals m , the number of schools), $\gamma(P_i, Q_{-i})R_i\gamma(Q_i, Q_{-i})$. Let $Q'_i := \gamma(P_i, Q_{-i})(i)$. Clearly, $Q'_i \in \mathcal{Q}(k)$. By Lemma A.1, $\gamma(Q'_i, Q_{-i})(i) = \gamma(P_i, Q_{-i})(i)$. Hence, $\gamma(Q'_i, Q_{-i})R_i\gamma(Q_i, Q_{-i})$. If $\gamma(Q'_i, Q_{-i})P_i\gamma(Q_i, Q_{-i})$, then $Q \notin \mathcal{E}^\gamma(P, k)$, a contradiction. Hence, $\gamma(Q'_i, Q_{-i})(i) = \gamma(Q_i, Q_{-i})(i)$.

By Lemma A.5, γ is non bossy. Hence, $\gamma(P_i, Q_{-i}) = \gamma(Q'_i, Q_{-i}) = \gamma(Q)$. In particular, $\gamma(P_i, Q_{-i})(j) = \gamma(Q)(j) = s$. Since $sP_i\gamma(Q)(i) = \gamma(P_i, Q_{-i})(i)$, student i has justified envy at $\gamma(P_i, Q_{-i})$, contradicting $\gamma(P_i, Q_{-i}) \in S(P_i, Q_{-i})$. Hence, $\gamma(Q) \in S(P)$. ■

Proof of Theorem 6.5 Proposition 6.1 implies that the game $\Gamma^\gamma(P, 1) = \Gamma^\beta(P, 1)$ implements $S(P)$ in Nash equilibria, i.e., $S(P) = \mathcal{O}^\gamma(P, 1)$. Theorem 5.3 implies that $S(P) = \mathcal{O}^\gamma(P, 1) \subseteq \mathcal{O}^\gamma(P, k)$. Now Lemmas A.3 and A.6 complete the proof. ■

We need the following 2 lemmas for the proof of Theorem 7.5.

Lemma A.7 *Let φ be a mechanism such that for some $1 \leq k \leq m$, $\mathcal{O}^\varphi(P, k) \subseteq NW(P) \cap IR(P)$. Suppose $Q \in \mathcal{E}^\varphi(P, k)$ with $\varphi(Q) \notin PE(P)$. Then, there exist $p \geq 2$, a set of students $C_I = \{i_1, \dots, i_p\}$ and a set of schools $C_S = \{s_1, \dots, s_p\}$ such that for each school $s \in C_S$, $|\varphi(Q)(s)| = q_s$ and for each $i_r \in C_I$, $s_r P_{i_r} s_{r+1} = \varphi(Q)(i_r)$, where $s_{p+1} = s_1$.*

Proof Similar to a part of Step 1 of (iv) \Rightarrow (i) in Ergin (2002, proof of Theorem 1). ■

Lemma A.8 *Let P be a school choice problem. Let $\mu \in S(P)$. Define $Q_i := \mu(i)$ for all $i \in I$. Then, $\gamma(Q) = \mu$ and $Q \in \mathcal{E}^\gamma(P, 1) \subseteq \mathcal{E}^\gamma(P, k)$ for all $2 \leq k \leq m$.*

Proof Follows immediately from Theorem 5.3 and Proposition 6.2. ■

Proof of Theorem 7.5 We show that (i) \Rightarrow (ii) \Rightarrow (i) \Rightarrow (iii) \Rightarrow (i) \Rightarrow (iv) \Rightarrow (i).

(i) \Rightarrow (ii): Suppose P is a school choice problem with $|S(P)| \geq 2$. Hence, there is a stable matching μ different from the student-optimal stable matching μ_I . By optimality of μ_I , for each student $i \in I$, $\mu_I R_i \mu$, and for at least one student $i \in I$, $\mu_I P_i \mu$. So, $\mu \notin PE(P)$. By Lemma A.8, there exists a profile $Q \in \mathcal{Q}^I(1)$ such that (a) for each student $i \in I$, $Q_i = \mu(i)$; (b) $\gamma(Q) = \mu$; and (c) $Q \in \mathcal{E}^\gamma(P, 1)$. By Lemmas A.2 and A.7 there exist a set of students $C_I = \{i_1, \dots, i_p\}$ and a set of schools $C_S = \{s_1, \dots, s_p\}$ such that for each $s \in C_S$, $|\gamma(Q)(s)| = q_s$, and for each $i_l \in C_I$, $s_l P_{i_l} s_{l+1} = \gamma(Q)(i_l)$. Note that since a student is assigned to at most one school, for any two schools $s, s' \in C_S$, $\gamma(Q)(s) \cap \gamma(Q)(s') = \emptyset$. For any two subsets $I', I'' \subseteq I$ with $I' \cap I'' = \emptyset$ and any school s we will write $f_s(I') < f_s(I'')$ to say that for all students $i' \in I'$ and $i'' \in I''$, $f_s(i') < f_s(i'')$.

Step 1 *For each student $i_l \in C_I$, $i_l \in \gamma(Q)(s_{l+1})$ and $f_{s_l}(\gamma(Q)(s_l)) < f_{s_l}(i_l)$.*

By construction, $i_l \in \gamma(Q)(s_{l+1})$. Let $Q' = (s_l, Q_{-i_l})$. Since $Q \in \mathcal{E}^\gamma(P, 1)$, $\gamma(Q')(i_l) = i_l$. In particular, $f_{s_l}(\gamma(Q')(s_l)) < f_{s_l}(i_l)$. By (a), $\gamma(Q')(s_l) = \gamma(Q)(s_l)$, and Step 1 follows.

Step 2 *The priority structure f admits a weak X -cycle.*

From Step 1 it follows that the priority structure has the following form

f_{s_1}	f_{s_2}	\dots	$f_{s_{p-1}}$	f_{s_p}
\vdots	\vdots		\vdots	\vdots
$\gamma(Q)(s_1)$	$\gamma(Q)(s_2)$		$\gamma(Q)(s_{p-1})$	$\gamma(Q)(s_p)$
\vdots	\vdots		\vdots	\vdots
i_1	i_2		i_{p-1}	i_p
\vdots	\vdots		\vdots	\vdots

Suppose $f_{s_1}(i_2) < f_{s_1}(i_1)$. Since $i_1 \in \gamma(Q)(s_2)$, setting $i = i_2$, $i' = i_1$, $s = s_1$, $s' = s_2$, $I_s = \gamma(Q)(s_1) \setminus i_p$, and $I_{s'} = \gamma(Q)(s_2) \setminus i_1$ yields a weak X -cycle.

Suppose now $f_{s_1}(i_1) < f_{s_1}(i_2)$. If $f_{s_1}(i_3) < f_{s_1}(i_2)$, then we obtain again a weak X -cycle by setting $i = i_3$, $i' = i_2$, $s = s_1$, $s' = s_3$, $I_s = \gamma(Q)(s_1) \setminus i_p$, and $I_{s'} = \gamma(Q)(s_3) \setminus i_2$. So, suppose $f_{s_1}(i_2) < f_{s_1}(i_3)$. By repeating this reasoning with students i_4, i_5, \dots, i_{p-1} we either obtain a weak X -cycle or establish that the priority ordering of school s_1 has the following form:

$$f_{s_1} : \dots \gamma(Q)(s_1) \dots i_1 \dots i_2 \dots i_{p-2} \dots i_{p-1} \dots$$

To deal with the latter case, recall that by construction, $i_p \in \gamma(Q)(s_1)$ and $i_{p-1} \in \gamma(Q)(s_p)$. So, we obtain a weak X -cycle by setting $i = i_p$, $i' = i_{p-1}$, $s = s_1$, $s' = s_p$, $I_s = \gamma(Q)(s_1) \setminus i_p$, and $I_{s'} = \gamma(Q)(s_p) \setminus i_{p-1}$.

(ii) \Rightarrow (i): Without loss of generality, let students $i_1 = i$ and $i_2 = i'$ and schools $s_1 = s$ and $s_2 = s'$ be the agents involved in a weak X -cycle. Without loss of generality we may assume that $I_{s_1} = \{i_3, \dots, i_{q_{s_1}+1}\}$ and $I_{s_2} = \{i_{q_{s_1}+2}, \dots, i_{q_{s_1}+q_{s_2}}\}$. Consider the students' preferences P given below. (Unacceptable schools are not depicted.)

P_{i_1}	P_{i_2}	P_{i_3}	\dots	$P_{i_{q_{s_1}+1}}$	$P_{i_{q_{s_1}+2}}$	\dots	$P_{i_{q_{s_1}+q_{s_2}}}$	$P_{i_{q_{s_1}+q_{s_2}+1}}$	\dots	P_{i_n}
s_1	s_2	s_1	s_1	s_1	s_2	s_2	s_2	s_2	s_2	s_2
s_2	s_1									

Note that for $j = q_{s_1} + q_{s_2} + 1, \dots, n$, student i_j finds all schools unacceptable. One easily verifies that the distinct matchings

$$\mu_I = \left(\begin{array}{cccccccccc} i_1 & i_2 & i_3 & \dots & i_{q_{s_1}+1} & i_{q_{s_1}+2} & \dots & i_{q_{s_1}+q_{s_2}} & i_{q_{s_1}+q_{s_2}+1} & \dots & i_n \\ s_1 & s_2 & s_1 & \dots & s_1 & s_2 & \dots & s_2 & i_{q_{s_1}+q_{s_2}+1} & \dots & i_n \end{array} \right)$$

and

$$\mu_S = \left(\begin{array}{cccccccccc} i_1 & i_2 & i_3 & \dots & i_{q_{s_1}+1} & i_{q_{s_1}+2} & \dots & i_{q_{s_1}+q_{s_2}} & i_{q_{s_1}+q_{s_2}+1} & \dots & i_n \\ s_2 & s_1 & s_1 & \dots & s_1 & s_2 & \dots & s_2 & i_{q_{s_1}+q_{s_2}+1} & \dots & i_n \end{array} \right)$$

are stable. So, $|S(P)| \geq 2$.

(i) \Rightarrow (iii): Let P be a school choice problem with $Q \in \mathcal{E}^\gamma(P, k)$ such that $\gamma(Q) \notin PE(P)$. Suppose first that $|S(P)| \geq 2$. Then, by (i) \Rightarrow (ii), f admits a weak X -cycle. So, suppose $|S(P)| = 1$. By Lemma A.8, there exists $\bar{Q} \in \mathcal{E}^\gamma(P, 1)$ with $\gamma(\bar{Q}) = \gamma(P) =: \mu_I$. If $\mu_I \notin PE(P)$ then by Theorem 1 of Ergin (2002) f admits an Ergin-cycle, which by

Remark 7.4 implies that f admits a weak X -cycle. So, suppose $\mu_I \in PE(P)$. Since $\gamma(Q) \notin PE(P)$, $\gamma(Q) \neq \mu_I$. Since $|S(P)| = 1$, $\gamma(Q) \notin S(P)$. By Theorem 6.5, f admits an Ergin-cycle, which by Remark 7.4 implies again that f admits a weak X -cycle.

(iii) \Rightarrow (i): Suppose that f admits a weak X -cycle. By (ii) \Rightarrow (i) there exists P with $|S(P)| \geq 2$. So, there exists $\mu \in S(P) \setminus PE(P)$. By Lemma A.8, there exists $Q \in \mathcal{E}^\gamma(P, k)$ with $\gamma(Q) = \mu$.

(i) \Rightarrow (iv): Let P be a school choice problem with $Q \in \mathcal{E}^\beta(P, k)$ such that $\beta(Q) \notin PE(P)$. By Theorem 6.1, $\beta(Q) \in S(P)$. Hence, by Lemma A.8, there exists $\bar{Q} \in \mathcal{E}^\gamma(P, 1)$ with $\gamma(\bar{Q}) = \beta(Q)$. Hence, from (i) \Rightarrow (iii) it follows that f admits a weak X -cycle.

(iv) \Rightarrow (i): Suppose that f admits a weak X -cycle. From (iii) \Rightarrow (i) it follows that there is a school choice problem P with $Q \in \mathcal{E}^\gamma(P, 1)$ such that $\gamma(Q) \notin PE(P)$. Since $\Gamma^\beta(P, 1) = \Gamma^\gamma(P, 1)$, the result follows. \blacksquare

B Appendix: Proofs for TTC

We first introduce some graph-theoretic notation to provide concise proofs of our results. Let $Q \in \mathcal{Q}^I$. Suppose the TTC algorithm is applied to Q , which we will denote by $TTC(Q)$, and suppose it terminates in no less than l steps. We denote by $G(Q, l)$ the (directed) graph that corresponds to step l . In this graph, the set of vertices $V(Q, l)$ is the set of agents present in step l . For any $v \in V(Q, l)$ there is a (unique) directed edge in $G(Q, l)$ from v to some $v' \in V(Q, l)$ (possibly $v' = v$ if $v \in I$) if agent v points to agent v' , which will also be denoted by $e(Q, l, v) = v'$.

A path (from v_1 to v_p) in $G(Q, l)$ is an ordered list of agents (v_1, v_2, \dots, v_p) such that $v_r \in V(Q, l)$ for all $r = 1, \dots, p$ and each v_r points to v_{r+1} for all $r = 1, \dots, p - 1$. A self-cycle (i) of a student i is a degenerate path, *i.e.*, i points to himself in $G(Q, l)$. An agent $v' \in V(Q, l)$ is a follower of an agent $v \in V(Q, l)$ if there is a path from v to v' in $G(Q, l)$. The set of followers of v is denoted by $F(Q, l, v)$. An agent $v' \in V(Q, l)$ is a predecessor of an agent $v \in V(Q, l)$ if there is a path from v' to v in $G(Q, l)$. The set of predecessors of v is denoted by $P(Q, l, v)$. A cycle in $G(Q, l)$ is a path (v_1, v_2, \dots, v_p) such that also v_p points to v_1 . Note that a self-cycle is a special case of a cycle. With a slight abuse of notation we sometimes refer to a cycle as the corresponding non ordered set of involved agents. Finally, for $v \in I \cup S$, let $\sigma(Q, v)$ denote the step of the TTC algorithm at which agent v is removed.

Observation B.1 In the TTC algorithm, once a student points to a school it will keep on pointing to the school in subsequent steps until he is assigned to a seat at the school or until the school has no longer available seats. In other words, if $i \in V(Q, l) \cap I$ for some step l of $TTC(Q)$ and $e(Q, l, i) = s \in S$, then $e(Q, r, i) = s$ for all steps r with $l \leq r \leq \min\{\sigma(Q, i), \sigma(Q, s)\}$. Similarly, once a school points to a student it will keep on pointing to the student in subsequent steps until the student is assigned to a seat at this or some other school. In other words, if $s \in V(Q, l) \cap S$ for some step l of $TTC(Q)$ and $e(Q, l, s) = i \in I$, then $e(Q, r, s) = i$ for all steps r with $l \leq r \leq \sigma(Q, i)$.

We now proceed to establish some preliminary results and slightly technical lemmas to be able to prove Theorems 5.4 and 7.2. The proof of the next lemma is omitted.

Lemma B.2 For any school choice problem P and any quota k , $\mathcal{O}^\tau(P, k) \subseteq IR(P)$.

In order to avoid possible confusion we will sometimes use an additional superindex Q and write $q_s^{Q,r}$ instead of q_s^r .

Lemma B.3 Let $Q \in \mathcal{Q}^I$. Let $i \in I$ and $Q'_i \in \mathcal{Q}$. Define $Q' := (Q'_i, Q_{-i})$. Suppose $\tau(Q)(i) \neq \tau(Q')(i)$. Let $p := \sigma(Q, i)$, $p' := \sigma(Q', i)$, and $r := \min\{p, p'\}$. Then,

- (a) at steps $1, \dots, r-1$, the same cycles form in $TTC(Q)$ and $TTC(Q')$;
- (b) $i \in V(Q, r) = V(Q', r)$ and for each school $s \in V(Q, r) \cap S$, $q_s^{Q,r} = q_s^{Q',r}$;
- (c) $e(Q, r, v) = e(Q', r, v)$ for each agent $v \in V(Q, r) \setminus i$;
- (d) there is a cycle C with $i \in C$ in either $G(Q, r)$ or $G(Q', r)$ (but not both).²⁶

Proof Item (a) follows from the proof of a result in Abdulkadiroğlu and Sönmez (1999, Lemma 1) or, alternatively, Abdulkadiroğlu and Sönmez (2003, Lemma). As for Item (b), from the definition of r , $i \in V(Q, r) \cap V(Q', r)$. The remainder of Item (b) follows directly from Item (a). Item (c) follows from Item (b) and the fact that $Q'_j = Q_j$ for all $j \in I \setminus i$. As for Item (d), by definition of r , there is a cycle C with $i \in C$ in $G(Q, r)$ or $G(Q', r)$. From Item (c) and $\tau(Q)(i) \neq \tau(Q')(i)$, $e(Q, r, i) \neq e(Q', r, i)$. In particular, C is not a cycle in both $G(Q, r)$ and $G(Q', r)$. This proves Item (d). ■

Lemma B.4 Let φ be a mechanism such that for any $Q \in \mathcal{Q}^I$, any $i \in I$, $Q'_i = \varphi(Q)(i) \in \mathcal{Q}(1)$ implies $\varphi(Q'_i, Q_{-i}) = \varphi(Q)$. Then, for any school choice problem P and quotas $k < k'$, $\mathcal{E}^\varphi(P, k) \subseteq \mathcal{E}^\varphi(P, k')$.

²⁶Note that it is still possible that there is another cycle \bar{C} (i.e., $\bar{C} \neq C$) with $i \in \bar{C}$ present in the other graph.

Proof Let $Q \in \mathcal{E}^\varphi(P, k)$. Suppose $Q \notin \mathcal{E}^\varphi(P, k')$. Then, there is a student i with a strategy $\bar{Q}_i \in \mathcal{Q}(k')$ such that $\varphi(\bar{Q}_i, Q_{-i})P_i\varphi(Q)$. Let $\bar{Q}'_i := \varphi(\bar{Q}_i, Q_{-i})(i)$. Clearly, $\bar{Q}'_i \in \mathcal{Q}(k)$. By assumption, $\varphi(\bar{Q}'_i, Q_{-i}) = \varphi(\bar{Q}_i, Q_{-i})$. So, $\varphi(\bar{Q}'_i, Q_{-i})P_i\varphi(Q)$, contradicting $Q \in \mathcal{E}^\varphi(P, k)$. Hence, $Q \in \mathcal{E}^\varphi(P, k')$. \blacksquare

Proposition B.5 *For any $Q \in \mathcal{Q}^I$, any $i \in I$, $Q'_i = \tau(Q)(i) \in \mathcal{Q}(1)$ implies $\tau(Q'_i, Q_{-i}) = \tau(Q)$. In particular, for any school choice problem P and quotas $k < k'$, $\mathcal{E}^\tau(P, k) \subseteq \mathcal{E}^\tau(P, k')$.*

Proof Let $Q \in \mathcal{Q}^I$. Let $i \in I$ and define $Q'_i := \tau(Q)(i) \in \mathcal{Q}(1)$. Define $Q' := (Q'_i, Q_{-i})$. We have to show that $\tau(Q') = \tau(Q)$. By non bossiness of τ ,²⁷ it is sufficient to show that $\tau(Q')(i) = \tau(Q)(i)$. If $\tau(Q)(i) = i$, then from the definition of the TTC algorithm, $\tau(Q')(i) = i = \tau(Q)(i)$.

So, suppose $\tau(Q)(i) =: s \in S$. Suppose to the contrary that $\tau(Q')(i) \neq \tau(Q)(i)$. Then, since $Q'_i = \tau(Q)(i) = s$, student i remains unassigned under Q' , *i.e.*, $\tau(Q')(i) = i$. Let $p := \sigma(Q, i)$, $p' := \sigma(Q', i)$, and $r := \min\{p, p'\}$. By Lemma B.3(d), there is a cycle C with $i \in C$ in either $G(Q, r)$ or $G(Q', r)$ (but not both).

Suppose cycle C is in $G(Q, r)$ but not in $G(Q', r)$. Since student i is assigned through cycle C and $\tau(Q)(i) = s$, $e(Q, r, i) = s$. Since $e(Q', r, i) \neq e(Q, r, i)$ and $Q'_i = \tau(Q)(i) = s$, $e(Q', r, i) = i$. Hence, at the beginning of step r of $TTC(Q')$, school s has no available seats, *i.e.*, $q_s^{Q', r} = 0$. By Lemma B.3(b), $q_s^{Q, r} = q_s^{Q', r} = 0$. So, $e(Q, r, i) \neq s$, a contradiction.

So, cycle C is in $G(Q', r)$ but not in $G(Q, r)$. If $e(Q', r, i) = s$, then $\tau(Q')(i) = s$, a contradiction with $\tau(Q')(i) \neq \tau(Q)(i)$. So by $Q'_i = \tau(Q)(i) = s$, $e(Q', r, i) = i$, *i.e.*, $C = (i)$ is a self-cycle. Since $i \in V(Q, r)$ and $\tau(Q)(i) = s$, $q_s^{Q, r} > 0$. By Lemma B.3(b), $q_s^{Q', r} = q_s^{Q, r} > 0$. So, $s \in V(Q', r)$. But then from $Q'_i = s$, $e(Q', r, i) = s$, a contradiction. We conclude that $\tau(Q')(i) = \tau(Q)(i)$. \blacksquare

Lemma B.6 *Let $\bar{Q} \in \mathcal{Q}^I$. Let $v, v' \in I \cup S$, $v \neq v'$. Suppose $v' \in P(\bar{Q}, l, v)$ at some step l of $TTC(\bar{Q})$. Then, $\sigma(\bar{Q}, v) \leq \sigma(\bar{Q}, v')$ and $[\sigma(\bar{Q}, v) = \sigma(\bar{Q}, v')$ only if v and v' are removed in the same cycle].*

Proof By Observation B.1, each agent in the path from v' to v will keep on pointing to its (direct) follower at least until the step in which agent v is removed, *i.e.*, step $\sigma(\bar{Q}, v)$.

²⁷Pápai's (2000) main result implies that τ is group strategy-proof. Group strategy-proofness implies non bossiness.

Hence, $\sigma(\bar{Q}, v) \leq \sigma(\bar{Q}, v')$. Suppose $\sigma(\bar{Q}, v) = \sigma(\bar{Q}, v')$. Then, all agents in the path from v' to v form part of a cycle at this step. Since an agent can be part of at most one cycle at a given step, all agents in the path from v' to v are in the same cycle. ■

Lemma B.7 *Let $Q \in \mathcal{Q}^I$. Let $i \in I$ and $Q'_i \in \mathcal{Q}$. Define $Q' := (Q'_i, Q_{-i})$. Suppose $\tau(Q)(i) \neq \tau(Q')(i)$ and $\sigma(Q, i) \leq \sigma(Q', i)$. Then, for each step l with $\sigma(Q, i) \leq l \leq \sigma(Q', i)$, if $v \in V(Q', l) \setminus (P(Q', l, i) \cup i)$ then $v \in V(Q, l)$ and $F(Q, l, v) = F(Q', l, v)$.²⁸*

Proof Let $p := \sigma(Q, i)$ and $p' := \sigma(Q', i)$. From Lemma B.3(b),

$$V(Q, p) = V(Q', p) \text{ and } q_s^{Q,p} = q_s^{Q',p} \text{ for each school } s \in V(Q, p) \cap S. \quad (1)$$

With a slight abuse of notation, for each $l, p \leq l \leq p'$, denote $P_l = P(Q', l, i) \cup i$. From Observation B.1,

$$P_p \subseteq P_{p+1} \subseteq \dots \subseteq P_{p'-1} \subseteq P_{p'}. \quad (2)$$

Also note

$$V(Q', p') \subseteq V(Q', p' - 1) \subseteq \dots \subseteq V(Q', p + 1) \subseteq V(Q', p). \quad (3)$$

We are done if we prove the following Claim(l) for each $l, p \leq l \leq p'$.

CLAIM(l): If $v \in V(Q', l) \setminus P_l$, then $v \in V(Q, l)$ and $e(Q, l, v) = e(Q', l, v)$.

Indeed, Claim(l) immediately implies the following Consequence(l):

CONSEQUENCE(l): If $v \in V(Q', l) \setminus P_l$, then $v \in V(Q, l)$ and $F(Q, l, v) = F(Q', l, v)$.

We now prove by induction that Claim(l) is true for each $l, p \leq l \leq p'$. By Lemma B.3(b,c), $V(Q, p) = V(Q', p)$ and $e(Q, p, v) = e(Q', p, v)$ for each agent $v \in V(Q, p) \setminus i$. Hence, Claim(p) is true.

If $p' = p$ we are done. So, suppose $p' \neq p$. Let l be a step such that $p < l \leq p'$. Assume Claim(g) is true for all $g, p \leq g < l \leq p'$. We prove that Claim(l) is true. Let $v \in V(Q', l) \setminus P_l$. From (2) and (3), $v \in V(Q', g) \setminus P_g$ for each step $g, p \leq g < l$. From Consequence(g) ($p \leq g < l$), $v \in V(Q, g)$ and

$$F(Q, g, v) = F(Q', g, v) \text{ for each step } g, p \leq g < l. \quad (4)$$

Since $v \in V(Q', l)$, v is not removed at the end of step $l - 1$ in $TTC(Q')$. Then by (1) and (4), v is also not removed at the end of step $l - 1$ in $TTC(Q)$. Hence, $v \in V(Q, l)$.

²⁸It follows immediately from the proof that the directed paths associated with $F(Q, l, v)$ and $F(Q', l, v)$ in $V(Q, l)$ and $V(Q', l)$, respectively, also coincide.

Assume Claim(l) is not true, *i.e.*, $x := e(Q, l, v) \neq e(Q', l, v) =: x'$. Since $v \notin P_l$, $x' \notin P_l$. By (2), $x' \notin P_{l-1}$. By (3) and $x' \in V(Q', l)$, $x' \in V(Q', l-1)$. By Consequence($l-1$), $x' \in V(Q, l-1)$. We distinguish between two cases.

CASE 1: Agent x' is removed at the end of step $l-1$ in $TTC(Q)$.

From (2) and (3), $x' \in V(Q', g) \setminus P_g$ for each step g , $p \leq g < l$. From Consequence(g) ($p \leq g < l$), $x' \in V(Q, g)$ and

$$F(Q, g, x') = F(Q', g, x') \text{ for each step } g, p \leq g < l. \quad (5)$$

By (1), (5), and the fact that x' is removed at the end of step $l-1$ in $TTC(Q)$, x' is also removed at the end of step $l-1$ in $TTC(Q')$. Hence, $x' \notin V(Q', l)$, a contradiction with $e(Q', l, v) = x'$.

CASE 2: Agent x' is *not* removed at the end of step $l-1$ in $TTC(Q)$.

Then, $x' \in V(Q, l)$. Since $e(Q, l, v) = x \neq x'$, we have $xQ_v x'$ (if v is a student) or $f_v(x) < f_v(x')$ (if v is a school). Hence, since $e(Q', l, v) = x'$, $x \notin V(Q', l)$. So, agent x was removed at some step g^* , $1 \leq g^* \leq l-1$, in $TTC(Q')$. In fact, by (1), $p \leq g^* \leq l-1$. Note that no agent in $P_{p'}$ is removed before the end of step p' in $TTC(Q')$. So, $x \notin P_{p'}$. By (2), $x \notin P_{g^*}$. Hence, $x \in V(Q', g^*) \setminus P_{g^*}$. By an argument similar to that of Case 1, x is also removed at the end of step g^* in $TTC(Q)$. Hence, $x \notin V(Q, l)$, a contradiction with $e(Q, l, v) = x$. ■

Lemma B.8 *Let $Q \in \mathcal{Q}^I$. Let $i \in I$ and $Q'_i \in \mathcal{Q}$. Define $Q' := (Q'_i, Q_{-i})$. Suppose there is a student $j \in I \setminus i$ with $\tau(Q)(j) \neq \tau(Q')(j)$. Then,*

- (a) $\sigma(Q, i) \leq \sigma(Q, j)$ and $[\sigma(Q, i) = \sigma(Q, j)$ only if i and j are assigned in the same cycle in $TTC(Q)$], and
- (b) $\sigma(Q', i) \leq \sigma(Q', j)$ and $[\sigma(Q', i) = \sigma(Q', j)$ only if i and j are assigned in the same cycle in $TTC(Q')$].

Proof By non bossiness of τ , $\tau(Q)(i) \neq \tau(Q')(i)$. Let $p := \sigma(Q, i)$ and $p' := \sigma(Q', i)$. Assume, without loss of generality, $p \leq p'$. Then, by definition of p and Lemma B.3(d), there is a cycle C in $G(Q, p)$ with $i \in C$ but not present in $G(Q', p)$.

We first prove (a). By Lemma B.3(a,b), for each student $h \in I \setminus i$ with $\sigma(Q, h) < p$ or $\sigma(Q', h) < p$, $\tau(Q)(h) = \tau(Q')(h)$. Let $r := \sigma(Q, j)$ and $r' := \sigma(Q', j)$. Since $\tau(Q)(j) \neq \tau(Q')(j)$, we have $r, r' \geq p$. So, $\sigma(Q, i) = p \leq r = \sigma(Q, j)$. Suppose $\sigma(Q, i) = \sigma(Q, j)$. We have to show that $j \in C$. Suppose $j \notin C$. Then, $j \in C^*$ for some cycle C^* , $C^* \neq C$, in $G(Q, p)$. Note $i \notin C^*$. By Lemma B.3(b), $V(Q, p) = V(Q', p)$. Hence, since

$e(Q, p, v) = e(Q', p, v)$ for each agent $v \in V(Q, p) \setminus i$, C^* is also a cycle in $G(Q', p)$. In particular, $\tau(Q)(j) = \tau(Q')(j)$, a contradiction. This completes the proof of (a).

We now prove (b). We distinguish between two cases.

CASE 1: $j \in P(Q', p, i)$.

Then, (b) follows directly from Lemma B.6 with $\bar{Q} = Q'$, $v' = j$, and $v = i$.

CASE 2: $j \notin P(Q', p, i)$.

Assume that (b) is not true. In other words, assume that $\sigma(Q', i) > \sigma(Q', j)$ or $[\sigma(Q', i) = \sigma(Q', j)$ and i and j are assigned in different cycles in $TTC(Q')$]. Then, $\sigma(Q, i) = p \leq r' = \sigma(Q', j) \leq \sigma(Q', i)$.

Note that by definition of r' , $j \in V(Q', r')$. Suppose $j \in (P(Q', r', i) \cup i)$. Since $j \neq i$, $j \in P(Q', r', i)$. By Lemma B.6, $\sigma(Q', i) \leq \sigma(Q', j)$ and $[\sigma(Q', i) = \sigma(Q', j)$ only if i and j are removed in the same cycle in $TTC(Q')$]. This contradicts the assumption that (b) is not true. So, $j \notin (P(Q', r', i) \cup i)$. In other words, $j \in V(Q', r') \setminus (P(Q', r', i) \cup i)$. Hence, by Lemma B.7, $j \in V(Q, r')$ and $F(Q, r', j) = F(Q', r', j)$. Since $\sigma(Q', j) = r'$, student j forms part of a cycle, say C' , in $G(Q', r')$. Hence, $C' = F(Q', r', j)$. So, also $C' = F(Q, r', j)$. Hence, student j is assigned to the same school (or himself) in $TTC(Q)$ and $TTC(Q')$, contradicting $\tau(Q)(j) \neq \tau(Q')(j)$. This completes the proof of (b). ■

Proposition B.9 *Let P be a school choice problem. Let $2 \leq k \leq m$. Let $Q \in \mathcal{E}^\tau(P, k)$. Define $\bar{Q}_i := \tau(Q)(i)$ for all $i \in I$. Then, $\bar{Q} \in \mathcal{E}^\tau(P, 1)$ and $\tau(\bar{Q}) = \tau(Q)$. In particular, $\mathcal{O}^\tau(P, k) \subseteq \mathcal{O}^\tau(P, 1)$.*

Proof It is sufficient to prove the following claim.

CLAIM: Let P be a school choice problem. Let $2 \leq k \leq m$, $Q \in \mathcal{E}^\tau(P, k)$, and $j \in I$. Let $\tilde{Q}_j := \tau(Q)(j)$. Then, $\tilde{Q} := (\tilde{Q}_j, Q_{-j}) \in \mathcal{E}^\tau(P, k)$.

Indeed, if the Claim is true then we can pick students one after another and each time apply both the Claim and Proposition B.5 to eventually obtain a profile $\bar{Q} \in \mathcal{E}^\tau(P, k)$ with $\tau(\bar{Q}) = \tau(Q)$ and where for all $j \in I$, $\bar{Q}_j = \tau(Q)(j)$. By construction, $\bar{Q} \in \mathcal{Q}(1)^I$. So, $\bar{Q} \in \mathcal{E}^\tau(P, 1)$.

To prove the Claim, suppose $\tilde{Q} \notin \mathcal{E}^\tau(P, k)$. Then, there is a student i with a profitable deviation at \tilde{Q} in $\Gamma^\tau(P, k)$. In fact, by Proposition B.5 there is a strategy $Q'_i \in \mathcal{Q}(1)$ with

$$\tau(Q'_i, \tilde{Q}_{-i}) P_i \tau(Q_i, \tilde{Q}_{-i}). \quad (6)$$

We claim $i \neq j$. Suppose $i = j$. Then, $\tilde{Q}_{-i} = \tilde{Q}_{-j} = Q_{-j}$. So, (6) becomes $\tau(Q'_i, Q_{-j}) P_j \tau(Q_j, Q_{-j})$, contradicting $Q \in \mathcal{E}^\tau(P, k)$. So, $i \neq j$.

Recall $\tilde{Q} = (Q_i, \tilde{Q}_j, Q_{-ij})$. Define $\tilde{Q}' := (Q'_i, \tilde{Q}_j, Q_{-ij})$ and $Q' := (Q'_i, Q_j, Q_{-ij})$. We can rewrite (6) as

$$\tau(\tilde{Q}') = \tau(Q'_i, \tilde{Q}_j, Q_{-ij})P_i\tau(Q_i, \tilde{Q}_j, Q_{-ij}) = \tau(\tilde{Q}). \quad (7)$$

From $Q \in \mathcal{E}^\tau(P, k)$, Lemma B.2 ($\mathcal{O}^\tau(P, k) \subseteq IR(P)$), and Proposition B.5, $\tau(\tilde{Q}) = \tau(Q) \in IR(P)$. By (7), $\tau(\tilde{Q}')(i) =: s \in S$. Since $\tilde{Q}'_i = Q'_i \in \mathcal{Q}(1)$, $Q'_i = s$.

Suppose $\tau(Q')(j) = \tau(Q)(j)$. Recall $\tilde{Q}_j = \tau(Q)(j)$. So, $\tilde{Q}_j = \tau(Q')(j)$. Hence, Proposition B.5 implies $\tau(Q'_i, \tilde{Q}_j, Q_{-ij}) = \tau(Q'_i, Q_j, Q_{-ij})$ and $\tau(Q_i, \tilde{Q}_j, Q_{-ij}) = \tau(Q_i, Q_j, Q_{-ij})$. Then (7) can be rewritten as $\tau(Q'_i, Q_j, Q_{-ij})P_i\tau(Q_i, Q_j, Q_{-ij})$. So, $Q \notin \mathcal{E}^\tau(P, k)$, a contradiction. Hence, $\tau(Q')(j) \neq \tau(Q)(j)$.

Next, we prove that $\tau(Q')(i) \neq \tau(\tilde{Q}')(i)$. Suppose $\tau(Q')(i) = \tau(\tilde{Q}')(i)$. Since $\tau(\tilde{Q}) = \tau(Q)$, (7) boils down to $\tau(Q')P_i\tau(Q)$, which implies that $Q \notin \mathcal{E}^\tau(P, k)$, a contradiction. So, $\tau(Q')(i) \neq \tau(\tilde{Q}')(i)$.

Note that for any student $h \neq i$, $Q'_h = Q_h$. So, by Lemma B.8, $\sigma(Q', i) \leq \sigma(Q', j)$. Note also that for any student $h \neq j$, $\tilde{Q}'_h = Q'_h$. So, by Lemma B.8, $\sigma(Q', j) \leq \sigma(Q', i)$. So, $\sigma(Q', i) = \sigma(Q', j)$. From Lemma B.8 it follows that i and j are in the same cycle in $TTC(Q')$. So, i is not in a self-cycle. Hence, i is assigned to a school in $TTC(Q')$. Since $Q'_i = s$, $\tau(Q')(i) = s$. By definition, $s = \tau(\tilde{Q}')(i)$. So, $\tau(Q')(i) = \tau(\tilde{Q}')(i)$, a contradiction. Hence, $\tilde{Q} \in \mathcal{E}^\tau(P, k)$, which completes the proof of the Claim. \blacksquare

Proof of Theorem 5.4 Follows from Propositions B.5 and B.9. \blacksquare

In order to prove Theorem 6.8 we need the following two lemmas.

Lemma B.10 *Let f be a Kesten-cyclic priority structure. Let $1 \leq k \leq m$. Then, there is a school choice problem P with an unstable equilibrium outcome in the game $\Gamma^\tau(P, k)$, i.e., for some $Q \in \mathcal{E}^\tau(P, k)$, $\tau(Q) \notin S(P)$.*

Proof By Theorem 1 of Kesten (2006), there is a school choice problem P such that $\tau(P)$ is unstable. Since τ is strategy-proof, $P \in \mathcal{E}^\tau(P, m)$. Hence, by Theorem 5.4, $\tau(P) \in \mathcal{O}^\tau(P, m) = \mathcal{O}^\tau(P, k)$. So, there is a profile $Q \in \mathcal{Q}(k)^I$ such that $Q \in \mathcal{E}^\tau(P, k)$ and $\tau(Q) = \tau(P) \notin S(P)$. \blacksquare

Lemma B.11 *Let f be a Kesten-acyclic priority structure. Let $1 \leq k \leq m$. Then, for any school choice problem P all equilibrium outcomes in the game $\Gamma^\tau(P, k)$ are stable, i.e., for all $Q \in \mathcal{E}^\tau(P, k)$, $\tau(Q) \in S(P)$. In fact, $S(P) = \mathcal{O}^\tau(P, k)$.*

Proof By Theorem 1 of Kesten (2006), $\tau = \gamma$. Hence, $\mathcal{O}^\tau(P, k) = \mathcal{O}^\gamma(P, k)$. By Lemma 1 of Kesten (2006), f is Ergin-acyclic. So, from Theorem 6.5, $S(P) = \mathcal{O}^\gamma(P, k) = \mathcal{O}^\tau(P, k)$. ■

Proof of Theorem 6.8 Follows from Lemmas B.10 and B.11. ■

Lemma B.12 *Let the priority structure f admit an X -cycle. Let $1 \leq k \leq m$. Then, there is a school choice problem P with a Pareto inefficient equilibrium outcome in the game $\Gamma^\tau(P, k)$, i.e., for some $Q \in \mathcal{E}^\tau(P, k)$, $\tau(Q) \notin PE(P)$.*

Proof Since f admits an X -cycle, we may assume, without loss of generality, that

- (a) $f_{s_1}(i_j) < f_{s_1}(i_1) < f_{s_1}(i_2)$ for each $j \in I_1 := \{3, \dots, q_{s_1} + 1\}$ and
- (b) $f_{s_2}(i_j) < f_{s_2}(i_2) < f_{s_2}(i_1)$ for each $j \in I_2 := \{q_{s_1} + 2, \dots, q_{s_1} + q_{s_2}\}$.

Consider students' preferences P defined by $P_{i_1} := s_2, s_1$, $P_{i_2} := s_1, s_2$, $P_{i_j} := s_1$ for $j \in I_1$, $P_{i_j} := s_2$ for $j \in I_2$, and $P_{i_j} := \emptyset$ for all $j \in \{q_{s_1} + q_{s_2} + 1, \dots, n\}$.

Consider $Q \in \mathcal{Q}(k)^I$ defined by $Q_{i_1} := s_1$, $Q_{i_2} := s_2$, and $Q_i := P_i$ for all $i \in I \setminus \{i_1, i_2\}$. One easily verifies that at $\tau(Q)$ all students in $\{i_3, i_4, \dots, i_{q_{s_1} + q_{s_2}}\}$ are assigned to their favorite school. Also, $\tau(Q)(i_1) = s_1$ and $\tau(Q)(i_2) = s_2$. It is obvious that at $\tau(Q)$ students i_1 and i_2 would like to swap their seats, i.e., $\tau(Q) \notin PE(P)$. Nevertheless, there is no unilateral deviation for either of the two students to obtain the other seat. Hence, $Q \in \mathcal{E}^\tau(P, k)$. ■

Proposition B.13 *Let $1 \leq k \leq m$. If for some school choice problem P there exists $Q \in \mathcal{E}^\tau(P, k)$ such that $\tau(Q) \notin PE(P)$ then f admits an X -cycle.*

Proof Let $Q \in \mathcal{E}^\tau(P, k)$ be such that $\tau(Q) \notin PE(P)$. In view of Proposition B.9 we may assume without loss of generality that $k = 1$ and for each student $i \in I$, $Q_i = \tau(Q)(i)$. For any school $s \in S$ and any profile $\widehat{Q} \in \mathcal{Q}^I$, let $A_s(\widehat{Q})$ be the set of students to which school s points whenever school s is part of a cycle, i.e.,

$$A_s(\widehat{Q}) := \{i \in I : \text{there is a step } l \text{ of } TTC(\widehat{Q}) \text{ with } i = e(\widehat{Q}, l, s) \text{ and } s \in F(\widehat{Q}, l, i)\}.$$

Step 1 *There exist $p \geq 2$, a set of students $C_I = \{i_1, \dots, i_p\}$, and a set of schools $C_S = \{s_1, \dots, s_p\}$ such that*

- (a) *for each student $i_r \in C_I$, $s_r P_{i_r} s_{r+1} = \tau(Q)(i_r)$ (where $i_{p+1} = i_0$),*
- (b) *for each school $s \in C_S$, $|A_s(Q)| = |\tau(Q)(s)| = q_s$, and*
- (c) *for any two distinct schools $s, s' \in C_S$, $A_s(Q) \cap A_{s'}(Q) = \emptyset$.*

One easily shows that that $\mathcal{O}^\tau(P, 1) \subseteq NW(P) \cap IR(P)$. Hence, by Lemma A.7 there exist a set of students $C_I = \{i_1, \dots, i_p\}$ and a set of schools $C_S = \{s_1, \dots, s_p\}$ such that for each $i_r \in C_I$, $s_r P_{i_r} s_{r+1} = \tau(Q)(i_r)$. This proves (a). Obviously,

$$|A_s(\widehat{Q})| = |\tau(\widehat{Q})(s)| \text{ for any } \widehat{Q} \in \mathcal{Q}^I. \quad (8)$$

Hence, by Lemma A.7, for each school $s \in C_S$, $|A_s(Q)| = |\tau(Q)(s)| = q_s$. Hence, (b) follows. Since a student is part of exactly 1 cycle, (c) follows. \square

For any student $i_r \in C_I$ define $Q^r := (s_r, Q_{-i_r})$. By Step 1, $\tau(Q)(i_r) = Q_{i_r} = s_{r+1}$ (modulo p). Since $Q_{i_r}^r = s_r P_{i_r} s_{r+1}$ and $Q \in \mathcal{E}^\tau(P, 1)$, $\tau(Q^r)(i_r) = i_r$.

Step 2 *Let $r \in \{1, \dots, p\}$. For each $s \in S \setminus s_{r+1}$, $|A_s(Q^r)| = |A_s(Q)|$, and $|A_{s_{r+1}}(Q^r)| = |A_{s_{r+1}}(Q)| - 1$.*

In view of (8) and $i_r \in \tau(Q)(S_{r+1})$, we are done if we prove that for each $s \in S \setminus s_{r+1}$, $\tau(Q^r)(s) = \tau(Q)(s)$, and $\tau(Q^r)(s_{r+1}) = \tau(Q)(s_{r+1}) \setminus i_r$.

For each school $s \neq s_r, s_{r+1}$, only the q_s (see Step 1(b)) students in $\tau(Q)(s)$ list school s in Q^r . Note also that only the $q_{s_{r+1}} - 1$ (see Step 1(b)) students in $\tau(Q)(s_{r+1}) \setminus i_r$ list school s_{r+1} in Q^r . Since $Q^r \in \mathcal{Q}^I(1)$ and $\tau(Q^r) \in NW(Q^r)$, $\tau(Q^r)(s) = \tau(Q)(s)$ for each school $s \neq s_r$. Finally, note that only the $q_{s_r} + 1$ students in $\tau(Q)(s_r) \cup i_r$ list school s_r in Q^r . Since $Q_{i_r}^r = s_r$, $\tau(Q^r)(i_r) = i_r$, and $\tau(Q^r) \in NW(Q^r)$, $\tau(Q^r)(s_r) = \tau(Q)(s_r)$. \square

Step 3 *For any $r \in \{1, \dots, p\}$, $A_{s_r}(Q^r) = A_{s_r}(Q)$.*

We are done if we show the following claim.

CLAIM: *For each integer l , we have*

$$\left. \begin{array}{l} [s_r \in V(Q, l) \text{ if and only if } s_r \in V(Q^r, l)], \\ [\text{if } s_r \text{ is in a cycle } C \text{ of } G(Q, l) \text{ then } C \text{ is also a cycle of } G(Q^r, l)], \text{ and} \\ [\text{if } s_r \text{ is in a cycle } C \text{ of } G(Q^r, l) \text{ then } C \text{ is also a cycle of } G(Q, l)]. \end{array} \right\} \quad (9)$$

PROOF OF CLAIM: We distinguish among six cases.

CASE 1: $l < \min\{\sigma(Q, i_r), \sigma(Q^r, i_r)\}$. Then, (9) follows from Lemma B.3(a).

CASE 2: $\sigma(Q, i_r) \leq l < \sigma(Q^r, i_r)$. Since $Q_{i_r}^r = s_r$ and $\tau(Q^r)(i_r) = i_r$, $s_r \notin P(Q^r, l, i_r)$. Since $\sigma(Q^r, s_r) = \sigma(Q^r, i_r) - 1$, $s_r \in V(Q^r, l)$. By Lemma B.7, $s_r \in V(Q, l)$ and $F(Q, l, s_r) = F(Q^r, l, s_r)$ (as directed paths). So, (9) holds.

CASE 3: $\sigma(Q, i_r) < l = \sigma(Q^r, i_r)$. Since $\sigma(Q^r, i_r) = \sigma(Q^r, s_r) + 1$, $F(Q^r, l - 1, s_r)$ is the last cycle of s_r under $TTC(Q^r)$. By Case 2, this is a cycle of s_r under $TTC(Q)$. In fact,

by Cases 1 and 2 and Step 2, this is also the last cycle of s_r under $TTC(Q)$. Hence, $s_r \notin V(Q, l)$ and $s_r \notin V(Q^r, l)$, and (9) holds trivially.

CASE 4: $\sigma(Q, i_r) < \sigma(Q^r, i_r) < l$. From the proof of Case 3, $s_r \notin V(Q, l)$ and $s_r \notin V(Q^r, l)$. Hence, (9) holds trivially.

CASE 5: $\sigma(Q^r, i_r) = l \leq \sigma(Q, i_r)$. Since $\sigma(Q^r, i_r) = \sigma(Q^r, s_r) + 1$, $F(Q^r, l - 1, s_r)$ is the last cycle of s_r under $TTC(Q^r)$. By Case 1, this is a cycle of s_r under $TTC(Q)$. In fact, by Case 1 and Step 2, this is also the last cycle of s_r under $TTC(Q)$. Hence, $s_r \notin V(Q, l)$ and $s_r \notin V(Q^r, l)$, and (9) holds trivially.

CASE 6: $\sigma(Q^r, i_r) \leq \sigma(Q, i_r)$ and $l > \sigma(Q^r, i_r)$. From the proof of Case 5, $s_r \notin V(Q, l)$ and $s_r \notin V(Q^r, l)$. Hence, (9) holds trivially. \square

For each school $s_h \in C_S$, let j_h be the student to which school s_h points in the last cycle in which s_h appears under $TTC(Q)$, *i.e.*, $j_h := \operatorname{argmax}_{j \in A_{s_h}(Q)} f_{s_h}(j)$.

Step 4 For any $r \in \{1, \dots, p\}$, $f_{s_r}(j_r) < f_{s_r}(j_{r+1})$.

Suppose $f_{s_r}(j_r) > f_{s_r}(j_{r+1})$. From Step 3, $j_r \in A_{s_r}(Q) = A_{s_r}(Q^r)$ and in particular, $j_r = \operatorname{argmax}_{j \in A_{s_r}(Q^r)} f_{s_r}(j)$. So,

$$\sigma(Q^r, j_{r+1}) < \sigma(Q^r, j_r) = \sigma(Q^r, s_r) = \sigma(Q^r, i_r) - 1. \quad (10)$$

We will now prove the following claim to complete the proof of this step.

CLAIM: $\sigma(Q^r, i_r) \leq \sigma(Q^r, j_{r+1})$.

The claim yields a contradiction to (10). Hence, the assumption $f_{s_r}(j_r) > f_{s_r}(j_{r+1})$ is false. Note $j_r \neq j_{r+1}$. So, $f_{s_r}(j_r) < f_{s_r}(j_{r+1})$. \square

PROOF OF CLAIM:

Let j_{r+1}^* be the student to which school s_{r+1} points in i_r 's cycle under $TTC(Q)$, *i.e.*, $e(Q, \sigma(Q, i_r), s_{r+1}) = j_{r+1}^*$. We make the following two observations (O1 and O2).

O1. $\sigma(Q^r, j_{r+1}^*) \geq \sigma(Q^r, i_r)$.

[Proof: Suppose $\sigma(Q^r, j_{r+1}^*) < \sigma(Q^r, i_r)$.

Assume $\sigma(Q^r, i_r) \leq \sigma(Q, i_r)$. Denote $y^r = \sigma(Q^r, i_r)$. From Lemma B.3(a), $V(Q, y^r) = V(Q^r, y^r)$. By definition of j_{r+1}^* , $j_{r+1}^* \in V(Q, y^r)$. However, by assumption, $\sigma(Q^r, j_{r+1}^*) < y^r$, and hence, $j_{r+1}^* \notin V(Q^r, y^r) = V(Q, y^r)$, a contradiction.

So, $\sigma(Q, i_r) < \sigma(Q^r, i_r)$. Also note that $y := \sigma(Q, i_r) \leq \sigma(Q^r, j_{r+1}^*)$ (otherwise, by Lemma B.3(a), $j_{r+1}^* \notin V(Q, y)$, contradicting the definition of j_{r+1}^*). By Lemma

B.3(b,c), $j_{r+1}^* \in P(Q^r, y, i_r)$. Hence, $\sigma(Q^r, j_{r+1}^*) \geq y = \sigma(Q^r, i_r)$. Again a contradiction. So, $\sigma(Q^r, j_{r+1}^*) \geq \sigma(Q^r, i_r)$.]

O2. For all $i \in A_{s_{r+1}}(Q^r)$ with $f_{s_{r+1}}(j_{r+1}^*) \leq f_{s_{r+1}}(i)$, $\sigma(Q^r, i) \geq \sigma(Q^r, j_{r+1}^*)$.

Suppose $j_{r+1} \in A_{s_{r+1}}(Q^r)$. Note $f_{s_{r+1}}(j_{r+1}^*) \leq f_{s_{r+1}}(j_{r+1})$. From O2 (with $i = j_{r+1}$) and O1, $\sigma(Q^r, j_{r+1}) \geq \sigma(Q^r, j_{r+1}^*) \geq \sigma(Q^r, i_r)$.

Suppose now $j_{r+1} \notin A_{s_{r+1}}(Q^r)$. Assume $\sigma(Q^r, j_{r+1}) < \sigma(Q^r, i_r)$. Then,

$$j_{r+1} \neq i_r. \quad (11)$$

We consider two cases.

Case 1: $\sigma(Q^r, i_r) \leq \sigma(Q, i_r)$.

Then, $\sigma(Q^r, j_{r+1}) < \min\{\sigma(Q^r, i_r), \sigma(Q, i_r)\}$. From Lemma B.3(a) it follows that $\sigma(Q, j_{r+1}) = \sigma(Q^r, j_{r+1})$. So,

$$\sigma(Q, j_{r+1}) < \sigma(Q, i_r). \quad (12)$$

However, under $TTC(Q)$, i_r is in a cycle with s_{r+1} and j_{r+1} is in the last cycle of s_{r+1} . So, $\sigma(Q, j_{r+1}) \geq \sigma(Q, i_r)$, a contradiction to (12).

Case 2: $\sigma(Q, i_r) < \sigma(Q^r, i_r)$.

If $\sigma(Q^r, j_{r+1}) < \sigma(Q, i_r)$, then $\sigma(Q^r, j_{r+1}) < \min\{\sigma(Q^r, i_r), \sigma(Q, i_r)\}$ which yields the same contradiction as in Case 1. Therefore, $\sigma(Q, i_r) \leq \sigma(Q^r, j_{r+1})$.

From Observation B.1 and the assumption that $\sigma(Q^r, j_{r+1}) < \sigma(Q^r, i_r)$ it follows that for each $l \leq \sigma(Q^r, j_{r+1})$, $j_{r+1} \notin P(Q^r, l, i_r)$. From (11) and Lemma B.7 it follows that for each l with $\sigma(Q, i_r) \leq l \leq \sigma(Q^r, j_{r+1})$, $F(Q, l, j_{r+1}) = F(Q^r, l, j_{r+1})$ (as directed paths). So, by taking $l = \sigma(Q^r, j_{r+1})$, we obtain that j_{r+1} 's cycle under $TTC(Q)$ is the same as under $TTC(Q^r)$. In particular, $j_{r+1} \in A_{s_{r+1}}(Q^r)$, a contradiction.

Since both cases give a contradiction we conclude that $\sigma(Q^r, j_{r+1}) \geq \sigma(Q^r, i_r)$. \square

Step 5 *There is an X-cycle.*

We can assume, without loss of generality, that among the students in $\{j_1, \dots, j_p\}$ student j_1 is (one of) the last one(s) to be assigned to a school under $TTC(Q)$, *i.e.*,

$$\sigma(Q, j_1) \geq \sigma(Q, j_r) \text{ for any } r \in \{1, \dots, p\}. \quad (13)$$

Suppose $f_{s_2}(j_1) < f_{s_2}(j_2)$. By definition of j_2 , school s_2 points to student j_2 in the last, q_{s_2} -th, cycle of s_2 under $TTC(Q)$, which occurs at step $\sigma(Q, s_2)$. Hence, $j_1 \notin$

$V(Q, \sigma(Q, s_2))$. Hence, $\sigma(Q, j_1) < \sigma(Q, s_2) = \sigma(Q, j_2)$, contradicting (13). Since $j_1 \neq j_2$, $f_{s_2}(j_2) < f_{s_2}(j_1)$.

Let $s = s_1$, $s' = s_2$, $i = j_1$, and $i' = j_2$. We have just shown that $f_{s'}(i') < f_{s'}(i)$. By Step 4, $f_s(i) < f_s(i')$. Define $I_s := A_s(Q) \setminus i$ and $I_{s'} := A_{s'}(Q) \setminus i'$. By Step 1(b,c), I_s and $I_{s'}$ are disjoint sets such that $|I_s| = q_s - 1$ and $|I_{s'}| = q_{s'} - 1$. Moreover, by definition of $A_s(Q)$ and i , $I_s \subseteq U_s^f(i)$. Similarly, $I_{s'} \subseteq U_{s'}^f(i')$. Hence, schools s and s' together with students i and i' constitute an X -cycle. \square

Proof of Theorem 7.2 Follows from Lemma B.12 and Proposition B.13. \blacksquare

C Appendix: Proofs of Results in Section 8

Proof of Lemma 8.1 Let $\varphi := \gamma, \tau$. We will prove that $\varphi(P_i^k, Q_{-i})(i) = \varphi(Q)(i)$ for all $Q_{-i} \in \mathcal{Q}(k)^{I \setminus i}$ or $\varphi(P_i^k, Q'_{-i})P_i\varphi(Q_i, Q'_{-i})$ for some $Q'_{-i} \in \mathcal{Q}(k)^{I \setminus i}$. (This obviously completes the proof as it implies that no strategy k -dominates P_i^k .)

Suppose $\varphi(P_i^k, Q_{-i})(i) \neq \varphi(Q)(i)$ for some $Q_{-i} \in \mathcal{Q}(k)^{I \setminus i}$. We have to show that for some $Q'_{-i} \in \mathcal{Q}(k)^{I \setminus i}$, $\varphi(P_i^k, Q'_{-i})P_i\varphi(Q_i, Q'_{-i})$. Suppose that for some $\tilde{Q}_{-i} \in \mathcal{Q}^{I \setminus i}$, $\varphi(Q_i, \tilde{Q}_{-i})P_i\varphi(P_i^k, \tilde{Q}_{-i})$. Since $\varphi(P_i^k, \tilde{Q}_{-i})(i)R_i i$, we have $\tilde{s} := \varphi(Q_i, \tilde{Q}_{-i})(i) \in S$. From Lemma A.1 (for γ) and Lemma B.5 (for τ), $\varphi(\tilde{s}, \tilde{Q}_{-i})(i) = \varphi(Q_i, \tilde{Q}_{-i})(i)$.

Suppose \tilde{s} is also listed in P_i^k . Then,

$$\varphi(P_i^k, \tilde{Q}_{-i})(i)R_i\varphi(P_i^k, \tilde{Q}_{-i})(i) = \varphi(\tilde{s}, \tilde{Q}_{-i})(i) = \tilde{s}, \quad (14)$$

where P_i^k is the preference relation obtained from P_i^k by putting \tilde{s} in the first position. The first relation follows from Lemma 4.2. The second relation follows from the fact that the assignment by the DA/TTC algorithm does not change if a student makes more schools acceptable *and* puts them below the school he is assigned to. Clearly, (14) contradicts $\varphi(Q_i, \tilde{Q}_{-i})P_i\varphi(P_i^k, \tilde{Q}_{-i})$. Hence, \tilde{s} is not listed in P_i^k .

Let $\tilde{S} := \{s \in S \setminus \tilde{s} : sQ_i\tilde{s}\}$. The fact that \tilde{s} is not listed in P_i^k together with the definition of P_i^k implies that there is a school $s \notin \tilde{S}$ listed in P_i^k with $sP_i\tilde{s}$. Let s^* be the P_i -best school among the schools $s \notin \tilde{S}$ listed in P_i^k with $sP_i\tilde{s}$.

Suppose $\varphi = \gamma$. Since $\varphi(Q_i, \tilde{Q}_{-i}) \in NW(Q_i, \tilde{Q}_{-i})$, $|\varphi(Q_i, \tilde{Q}_{-i})(s)| = q_s$ for all $s \in \tilde{S}$. Clearly, for all $s \in \tilde{S}$, $i \notin \varphi(Q_i, \tilde{Q}_{-i})(s)$. Also, for all $s, t \in \tilde{S}$ with $s \neq t$, $\varphi(Q_i, \tilde{Q}_{-i})(s) \cap \varphi(Q_i, \tilde{Q}_{-i})(t) = \emptyset$. So we can define for $j \in I \setminus i$,

$$Q'_j := \begin{cases} s & \text{if } j \in \varphi(Q_i, \tilde{Q}_{-i})(s) \text{ for some } s \in \tilde{S}, \\ \emptyset & \text{otherwise.} \end{cases}$$

By the assumption that $\varphi = \gamma$, $\varphi(Q_i, \tilde{Q}_{-i}) \in S(Q_i, \tilde{Q}_{-i})$. Hence, $f_s(j) < f_s(i)$ for all $s \in \tilde{S}$ and all $j \in \varphi(Q_i, \tilde{Q}_{-i})(s)$. From $\varphi(Q_i, Q'_{-i}) \in S(Q_i, Q'_{-i})$ and the definition of Q'_{-i} , $\varphi(Q_i, Q'_{-i})(i) = \tilde{s}$. Similarly, $\varphi(P_i^k, Q'_{-i})(i) = s^*$. By definition of s^* , $\varphi(P_i^k, Q'_{-i})(i) = s^* P_i \tilde{s} = \varphi(Q_i, Q'_{-i})(i)$, which completes the proof for the case $\varphi = \gamma$.

Suppose $\varphi = \tau$. For any $s \in \tilde{S}$ define I_s as the set of students j that are assigned a seat through a cycle (in the TTC algorithm) of which school s is part and such that s points to j . Formally,

$$I_s := \left\{ j \in I : e \left((Q_i, \tilde{Q}_{-i}), \sigma[(Q_i, \tilde{Q}_{-i}), j], s \right) = j \in P \left((Q_i, \tilde{Q}_{-i}), \sigma[(Q_i, \tilde{Q}_{-i}), j], s \right) \right\}.$$

From Observation B.1 and $s Q_i \tilde{s} = \varphi(Q_i, \tilde{Q}_{-i})(i)$ for all $s \in \tilde{S}$ it follows that for all $s \in \tilde{S}$, $i \notin I_s$ and $|I_s| = q_s$. Also, for all $s, t \in \tilde{S}$ with $s \neq t$, $I_s \cap I_t = \emptyset$. So we can define for $j \in I \setminus i$,

$$Q'_j := \begin{cases} s & \text{if } j \in I_s \text{ for some } s \in \tilde{S}, \\ \emptyset & \text{otherwise.} \end{cases}$$

Since $s Q_i \tilde{s} = \varphi(Q_i, \tilde{Q}_{-i})(i)$ for all $s \in \tilde{S}$ it follows that $f_s(j) < f_s(i)$ for all $s \in \tilde{S}$ and all $j \in I_s$. From the definition of Q'_{-i} and the TTC algorithm, $\varphi(Q_i, Q'_{-i})(i) = \tilde{s}$. Similarly, $\varphi(P_i^k, Q'_{-i})(i) = s^*$, which completes the case $\varphi = \tau$ and hence the proof. ■

Proof of Proposition 8.5 By definition of the DA algorithm, $|M(\gamma(P^k))| \leq |M(\gamma(P))|$. We complete the proof by showing that if $i \in M(\gamma(P))$, then $\gamma(P^k) R_i \gamma(P)$. (Since $\gamma(P) \in IR(P)$, $\gamma(P^k)(i) \in S$. Hence, $i \in M(\gamma(P^k))$. But then $M(\gamma(P^k)) = M(\gamma(P))$.)

Let $i \in M(\gamma(P))$. Denote $s := \gamma(P)(i) \in S$. Suppose to the contrary that $s P_i \gamma(P^k)(i)$. Let $Q'_i := s$. By Lemma A.1, $\gamma(Q'_i, P_{-i})(i) = s$. By a result of Gale and Sotomayor (1985b, Theorem 2) extended to the college admissions model (Roth and Sotomayor, 1990, Theorem 5.34), Q'_i ranks $\gamma(Q'_i, P_{-i}^k)(i)$ weakly higher than $\gamma(Q'_i, P_{-i})(i)$. So, $\gamma(Q'_i, P_{-i}^k)(i) = s$, contradicting the assumption that $P^k \in \mathcal{E}(P, k)$. So, $\gamma(P^k)(i) R_i s = \gamma(P)(i)$. ■

Proof of Proposition 8.6 In Example 8.3, $\gamma(P) = \{\{i_1, s_1\}, \{i_2, s_3\}, \{i_3, s_2\}\}$ and $\tau(P) = \{\{i_1, s_1\}, \{i_3, s_3\}, \{i_2\}, \{s_2\}\}$. So, $|M(\tau(P))| = 2 < 3 = |M(\gamma(P))|$.

In Example 8.4, $\gamma(P) = \{\{i_2, s_3\}, \{i_3, s_2\}, \{i_1\}, \{s_1\}\}$ and $\tau(P) = \{\{i_1, s_2\}, \{i_2, s_3\}, \{i_3, s_1\}\}$. So, $|M(\tau(P))| = 3 > 2 = |M(\gamma(P))|$. ■

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