Chapter 3

Homogeneous Product Oligopoly Models

The logical approach to the study of models of economic interaction is to start by the static models of homogeneous product. Here we will find Cournot's model of which we will present a modern version. Quoting Shapiro (1989, p.333),

> Although a timeless model of oligopoly cannot, by definition, treat the essential issue of how rivals *react* to each other's actions, it does serve to elucidate the basic tension between competition and cooperation and provide an essential ingredient for the richer, dynamic analysis.

3.1 Cournot oligopoly. Quantity competition

Cournot proposes a model where a limited number of firms compete in a homogeneous product market. Consumers are passive and represented by an aggregate (inverse) demand function. Firms decide independently a production level (given their technologies). The interaction of aggregate output and aggregate demand determines the market clearing price typically, by means of an auctioneer.

3.1.1 Assumptions

The assumptions of the model are the following:

- structural assumptions
  - (i) It is a static model.
  - (ii) The production technology of every firm is summarized in a cost function $C_i(q_i)$, where q_i denotes firm i's production output.

(iii) The industry faces an aggregate demand function $Q = F(p)$, where Q denotes the aggregate production level. This assumption implicitly implies, (a) that we are referring to a homogeneous product market, and (b) that there is a large number of consumers.

(iv) There are n firms in the industry. This is a fix number, i.e. there is neither entry in nor exit from the industry.

(v) The strategic variable of the firms is their production levels.

- behavioral assumptions

  (a) Firms aim at maximizing profits. In the decision process firms are aware of the interaction among them: each firm knows that its production decision depends on its expectation over the rivals' decisions. Also, every rival's decision depend of what each of them think all the other competitors will decide. All firms take simultaneously their respective production decision.

  (b) Consumers choose a consumption bundle to maximize their utility functions defined in terms of consumption goods in the economy.

- assumptions on the demand function, $Q = F(p)$ (**S1**).

  The demand function tells us how many units of the good consumers are willing to buy at any given price p. We assume that F is *continuous*, *continuously differentiable*, *monotone*, *strictly decreasing* and *cuts the axes*. These assumptions guarantee that there exists an inverse demand function $p = F^{-1}(Q) \stackrel{\text{def}}{=} f(Q)$. Given that firms decide upon production levels, it is convenient to use the inverse demand function. It tells us the price at which consumers are willing to purchase any aggregate production level arriving in the market. Formally, the assumptions on the inverse demand function are:

  1. $f : \mathbf{R}_+ \to \mathbf{R}_+$
  2. $\exists \overline{Q} s.t. f(Q) \begin{cases} > 0 & \text{si } Q < \overline{Q}, \\ = 0 & \text{si } Q \geq \overline{Q} \end{cases}$
  3. $f(0) = \overline{p} < \infty$
  4. $f(Q)$ is continuous and differentiable in $[0, \overline{Q}]$
  5. $f'(Q) < 0 \;\text{ si }\; Q \in (0, \overline{Q})$

  In words, f is a real valued function (assumption 1), cutting the axes (assumptions 2 and 3), continuous and differentiable on the relevant domain (assumption 4), monotone and strictly decreasing (assumption 5). Alternatively to assumptions 2 and 3 we can assume that the total revenue of the industry is bounded above: $Qf(Q) \leq M < \infty$.

The aim of this set of assumptions is to define a compact set over which firms take decisions, thus guaranteeing the existence of a maximum.

- assumptions on the technology, $C_i(q_i)$ (**S2**).

  The production function of each firm is given and the factor markets are perfectly competitive. This implies that all the relevant information is embodied in the cost function. Each firm's cost function $C_i(q_i)$ is assumed *continuous, differentiable, strictly positive, with a non negative fixed cost* and *strictly increasing*. Formally,

  1. $C_i : \mathbf{R}_+ \to \mathbf{R}_+$
  2. C_i is continuous and differentiable $\forall q_i > 0$
  3. $C_i(q_i) > 0 \; \forall q_i > 0$
  4. $C_i(0) \geq 0$
  5. $C_i'(q_i) > 0 \; \forall q_i \geq 0$

- assumptions on the profit function, $\Pi_i(q)$ (**S3**).

  Let $q = (q_1, q_2, q_3, \dots, q_n)$ be a production plan. Firm i's profit function is defined as $\Pi_i(q) = q_i f(Q) - C_i(q_i)$ denotes firm i's profit function. A vector $\Pi(q) = (\Pi_1(q), \Pi_2(q), \Pi_3(q), \dots, \Pi_n(q))$ denotes a distribution of profits in the industry. Given the assumptions on the demand and cost functions, the profit function is *continuous, and differentiable*. Also, we assume that it is *strictly concave in* q_i. Formally,

  1. $\Pi_i : \mathbf{R}_+ \to \mathbf{R}_+$
  2. Π_i is $C^2 \; \forall q_i > 0$
  3. $\Pi_i(q)$ is strictly concave in $q_i, \; \forall q$ s.t. $q_i > 0, Q < \overline{Q}$.

As a consequence of assumptions **S1** and **S2**, the decision set of each firm is compact.

It should be clear that $q_i \in [0, \overline{Q}] \; \forall i$ because on the one hand, it is not possible to produce negative quantities and on the other hand, it does not make sense for a firm to produce above $\overline{Q}$ because $f(\overline{Q}) = 0$. Also, as $Q \to 0, \; p \to \overline{p}$, i.e. $\lim_{Q \to 0} = \overline{p}$. Hence, $\Pi_i(q)$ is defined on a compact set.

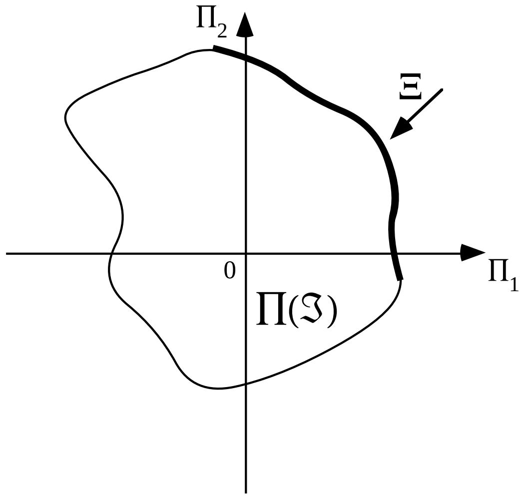

Figure 3.1: The space of outcomes.

Definition 3.1 (Feasibility)**.** *We say that* q_i *is a feasible output for firm* i *if* $q_i \in [0, \overline{Q}]$.

The set $\mathcal{F} \in \mathbf{R}^n$ *defined as* $\mathcal{F} \stackrel{def}{=} [0, \overline{Q}] \times [0, \overline{Q}] \times [0, \overline{Q}] \times \stackrel{n\ times}{\cdots} \times [0, \overline{Q}]$*, is the (compact) set of all feasible production plans in the industry.*

Definition 3.2 (Space of outcomes)**.** *The space of outcomes is the set of all possible distribution of profits in the industry:* $\{\Pi(q) | q \in \mathcal{F}\} \stackrel{def}{=} \Pi(\Im)$. *Obviously, it is also a compact set.*

Definition 3.3 (Pareto optimal outcomes)**.** *We say that an outcome* $\Pi(q)$ *is Pareto optimal if given a feasible production plan* q *associated to that outcome, any alternative feasible production plan* $q^{'}$ *generates a distribution of profits* $\Pi(q^{'})$ *that may allocate higher profits to some firms but not to all of them. Formally,* $PO = \{\Pi(q) | q \in \mathcal{F}\ s.t.\ \forall q' \in \mathcal{F},\ \Pi(q) > \Pi(q')\}$, *where* $\Pi(q) > \Pi(q')$ *means* $\Pi_i(q) \leq \Pi_i(q')\ \forall i$, *and* $\exists j\ s.t.\ \Pi_j(q) > \Pi_j(q')$.

Figure 3.1 illustrates these definitions for the two firms case.

3.1.2 Equilibrium.

Several equivalent definitions of the Cournot equilibrium are the following:

Definition 3.4. *A production plan* q^c *is a Cournot (Nash) equilibrium if there is no firm able to unilaterally improve upon its profit level modifying its production decision.*

Definition 3.5. *A production plan q^c is a Cournot (Nash) equilibrium if no firm has any profitable unilateral deviation.*

Definition 3.6. *Let $q^c_{-i} \stackrel{def}{=} (q^c_1, q^c_2, \ldots, q^c_{i-1}, q^c_{i+1}, \ldots, q^c_n)$. We say that a production plan $q^c \stackrel{def}{=} (q^c_1, q^c_2, \ldots, q^c_n)$ is a Cournot (Nash) equilibrium if $\Pi_i(q^c) = \max_{q_i} \Pi_i(q_i, q^c_{-i})\ \forall i$.*

Definition 3.7. *A production plan $q^c \stackrel{def}{=} (q^c_1, q^c_2, \ldots, q^c_n)$ is a Cournot (Nash) equilibrium if $\nexists \widetilde{q} \stackrel{def}{=} (q^c_1, q^c_2, \ldots, q^c_{i-1}, \widetilde{q}_i, q^c_{i+1}, \ldots, q^c_n)$ s.t. $\Pi(q^c) \leq \Pi(\widetilde{q})\ \forall i$.*

Definition 3.8. *A production plan $q^c \stackrel{def}{=} (q^c_1, q^c_2, \ldots, q^c_n)$ is a Cournot (Nash) equilibrium if $q^c_i = argmax_{q_i} \Pi_i(q_i, q^c_{-i})\ \forall i$, where $q^c_{-i} \stackrel{def}{=} (q^c_1, q^c_2, \ldots, q^c_{i-1}, q^c_{i+1}, \ldots, q^c_n)$.*

Given that firms decide upon their production levels simultaneously, any particular firm does not observe the actions of its rivals beforehand. The interpretation of the Nash non-cooperative equilibrium can be properly understood as every firm reasoning in the following way: "given that I think that my rivals will decide $q_{-i} \stackrel{\text{def}}{=} (q_1, q_2, \ldots, q_{i-1}, q_{i+1}, \ldots, q_n)$, my profit maximizing decision is q_i". The equilibrium arises when the expectations of all firms are fulfilled in the market. This is a fundamental equilibrium concept thus worth illustrating with an example.

Assume firm 1 forms the expectation that its competitors will decide the production plan $q^e_{-1} \stackrel{\text{def}}{=} (q^e_2, q^e_3, \ldots, q^e_n)$. Conditional on this expectation, its best decision will be $q^o_1 = \max_{q_1} \Pi_1(q_1, q^e_{-1})$. Next, firm 1 has to verify if every firm i would choose a production level q^e_i conditional on an expectation $q^e_{-i} \stackrel{\text{def}}{=} (q^o_1, q^e_2, \ldots, q^e_{i-1}, q^e_{i+1}, \ldots, q^e_n)$.

Let us assume that there is a firm i for which q^e_i is not the profit maximizing production level conditional on q^e_{-i}. Accordingly, firm 1 cannot expect that firm i will produce q^e_i. Therefore, the production plan $(q^o_1, q^e_2, \ldots, q^e_{i-1}, q^e_i, q^e_{i+1}, \ldots, q^e_n)$ cannot be a Cournot equilibrium.

If otherwise, for every firm i, q^e_i is the profit maximizing production level conditional on an expectation $(q^o_1, q^e_2, \ldots, q^e_{i-1}, q^e_{i+1}, \ldots, q^e_n)$, then all firms have consistent expectations. Therefore no one has any incentive to deviate producing a different output level from what its rivals expect. Then, a Cournot equilibrium is given by$(q^o_1, q^e_2, \ldots, q^e_{i-1}, q^e_i, q^e_{i+1}, \ldots, q^e_n)$.

In graphic terms we can envisage the Cournot equilibrium as every firm maximizing profits on the *expected* residual demand associated with the expectation of each firms on the behavior of the rival firms. Figure 3.2 illustrates the argument for two firms. When all firms' expectations are fulfilled we obtain the Cournot production plan.

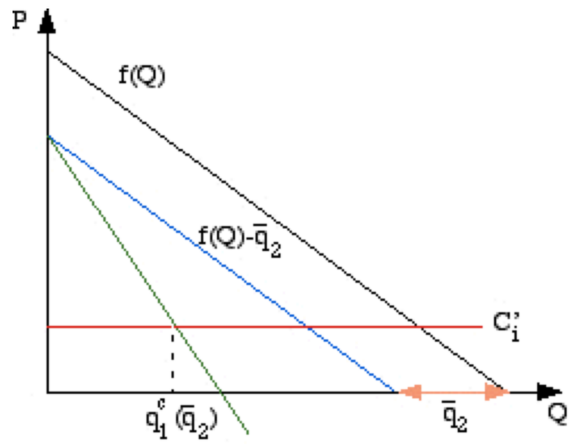

Figure 3.2: On the meaning of Cournot equilibrium

The description of the Cournot model can be reformulated in the jargon of game theory. There, the notion of Nash equilibrium can be directly linked to that of Cournot equilibrium.

In game theory terms, the Cournot model is a one-shot, simultaneous move, non-cooperative game. We consider only pure strategies. That is, each player chooses a simple action. We can represent this game in extensive form in figure 3.3 (See Martin (2002, p.42)) where, for the duopoly case, at decision node D_1, firm 1 chooses an output level q_1 from its strategy space $[0, \overline{Q}]$. At decision set D_2, firm 2, without knowing firm 1's decision, also chooses an output level q_2 from its strategy space $[0, \overline{Q}]$. This choice of output levels generate a distribution of profits $\Pi_1(q_1, q_2), \Pi_2(q_1, q_2)$.

We can also represent this game in normal form as a triplet $(N, \mathcal{F}, \pi)$, where $N = 1, 2, \ldots, n$ represents the set of players (firms), $\mathcal{F} \stackrel{\text{def}}{=} [0, \overline{Q}] \times \overset{\text{n times}}{\cdots} \times [0, \overline{Q}]$, is the strategy space, and Π is a vector of payoffs (i.e. a distribution of profits). Associated to this normal form representation, we have the payoff matrix given by table 3.1.

Then, we say that a vector of feasible actions (production plan), $(q_1, \ldots, q_n)$ is a Nash equilibrium if for all players and any feasible action q_i,

$$\Pi_i(q_i, q_{-i}^*) \leq \Pi_i(q_i^*, q_{-i}^*), \ \forall i. \tag{3.1}$$

3.1.3 Cournot equilibrium and Pareto optimality.

We will now examine whether the Cournot equilibrium satisfies the Pareto optimality property.

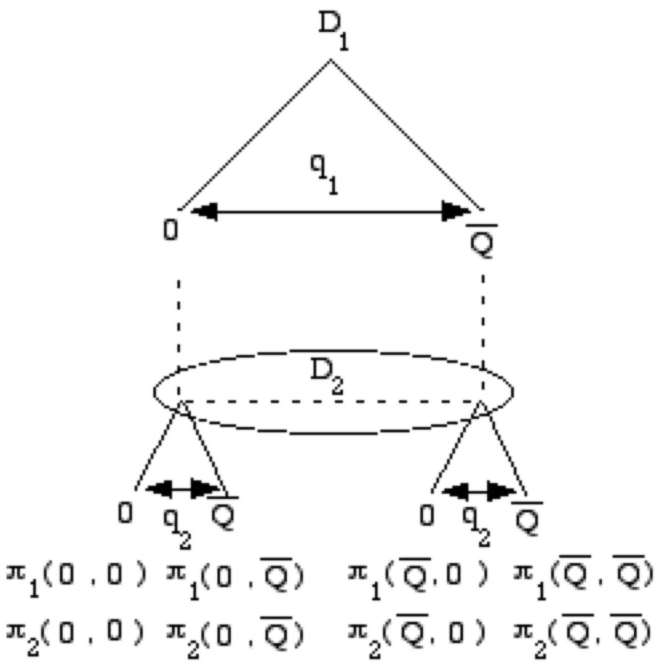

Figure 3.3: The Cournot model in extensive form.

Proposition 3.1. *Let $q^c \gg 0$ be a Cournot equilibrium production plan. Then $\Pi(q^c)$ is not Pareto optimum.*

Proof. The proof of this proposition is best developed in three steps.

(a) First, we will show that the aggregate production level in equilibrium is sold at a strictly positive price, i.e. $Q^c \stackrel{\text{def}}{=} \sum_i q_i^c < \overline{Q}$. Let us assume, a senso contrario, $Q^c \geq \overline{Q}$. Since $q^c \gg 0$ it satisfies the first order conditions of the profit maximization problem, that is $f(Q^c) + q_i^c f'(Q^c) \equiv C_i(q_i^c)$. But, $Q^c \geq \overline{Q}$ implies that this aggregate production level is sold at a zero price, $f(Q^c) = 0$, so that $q_i^c f'(Q^c) \equiv C_i(q_i^c)$ which is a contradiction.

(b) Next, given that $q_i^c > 0 \; \forall i$, the Cournot equilibrium is interior. This means that the set of first order conditions $f(Q) + q_i f'(Q) - C_i'(q_i) = 0$ characterize the equilibrium. Also, $\dfrac{\partial \Pi_i}{\partial q_j} = q_i f'(Q) < 0 \; \forall i, j \; i \neq j$. Hence a simultaneous reduction of the output levels of any two firms, q_i and q_j would improve their profits. Thus, we can find a production plan q such that $\Pi_i(q) > \Pi_i(q^c) \; \forall i$.

(c) Finally, the (simultaneous) reduction of output away from the equilibrium will have a negative effect on profits. Nevertheless, this is a second-order effect that normally is offset by the first-order effect described in (b). □

Intuitively, the conditions characterizing a Cournot equilibrium is the system

$firms$	0	$\dots q_1 \dots$	$\overline{Q}$
0	$\Pi_1(0,0), \Pi_2(0,0)$	$\Pi_1(q_1,0), \Pi_2(q_1,0)$	$\Pi_1(\overline{Q},0), \Pi_2(\overline{Q},0)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
q_2	$\Pi_1(0,q_2), \Pi_2(0,q_2)$	$\Pi_1(q_1,q_2), \Pi_2(q_1,q_2)$	$\Pi_1(\overline{Q},q_2), \Pi_2(\overline{Q},q_2)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\overline{Q}$	$\Pi_1(0,\overline{Q}), \Pi_2(0,\overline{Q})$	$\Pi_1(q_1,\overline{Q}), \Pi_2(q_2,\overline{Q})$	$\Pi_1(\overline{Q},\overline{Q}), \Pi_2(\overline{Q},\overline{Q})$

Table 3.1: Payoff matrix of the Cournot game.

of first order conditions associated to the profit maximization problem. That is, $f(Q^c)+q_i^c f'(Q^c)-C_i'(q_i^c) \equiv 0 \; \forall i$. This equation says that price is above marginal cost: $f(Q^c) > f(Q^c) + q_i^c f'(Q^c) = C_i'(q_i^c)$. Therefore, the optimality rule equating price and marginal cost is not satisfied. We should suspect that the Cournot equilibrium will not satisfy the property of Pareto optimality. This feature of the Cournot equilibrium means that firms have a way to improve their profit levels beyond the Cournot equilibrium level. One way of achieving these higher profits is by means of agreements. We will examine this topic in detail in chapter 4.

Alternatively, looking at the first order condition, we realize that it contains two elements. On the one hand, we have the difference between price and marginal cost $f(Q) - C_i'(q_i)$; on the other hand we have a term $q_i f'(Q)$ representing the effect of producing an additional unit. This effect consists in lowering the price in f' that affect all the units produced. These arguments illustrate the negative externality arising among firms. When firm i decides its production level takes into account the (adverse) effect of the price on its output (and thus on its profits) but ignores the effect on the aggregate production. Accordingly, every firm tends to produce at a level beyond what would be optimal at the industry level.

3.1.4 Existence of Cournot equilibrium.

An intuitive approach.

The Cournot equilibrium describes a situation where no firm has an incentive to reconsider its production decision conditional on the expectation on the rivals' decisions. Such an equilibrium does not necessarily exist. To illustrate the argument, following Okuguchi (1976) let us consider a duopoly where firm A has a lower cost than firm B. Figure 3.4 illustrates the scenario: DD' represents the market demand curve; DR represents the marginal revenue curve of one firm when the rival produces zero output; JJ' represents firm A's marginal cost; KK' represents firm B's marginal cost; KJ is the difference between marginal costs. This difference is independent of the production levels.

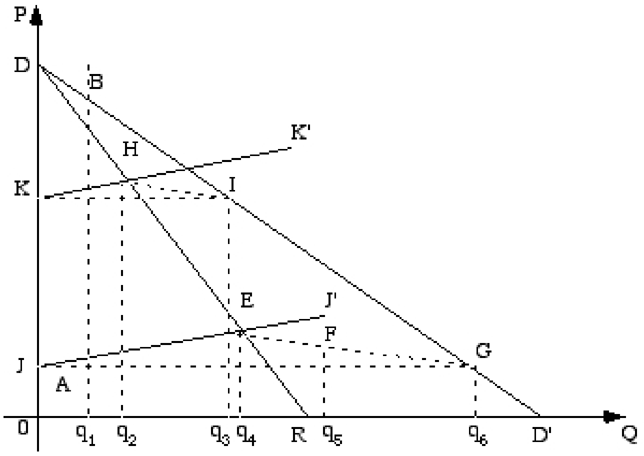

Figure 3.4: Existence of Cournot equilibrium (i).

- Assume that firm A expects firm B will produce $q_B = 0$. Then firm A's profit maximizing production level is $\widetilde{q}_A = q_4$ where marginal revenue equates firm A's marginal cost.

- Assume now that firm A expects firm B will produce $q_B = q_1$. The residual demand for firm A is BD'. This residual demand has its associated marginal revenue with origin at B (see figure 3.5). Firm A maintains its marginal cost JJ'. Therefore, $\widetilde{q}_A = q_1q_5$. The marginal revenue (= marginal cost) corresponding to the output q_1q_5 is given by q_5F. The aggregate production level is q_5.

- Similarly, if firm A has different conjectures on firm B'output, the point where marginal revenue equates marginal cost will move along the line EG in figure 3.4 and aggregate production increases from q_1 until q_6. Note that the production of firm B cannot go beyond q_6 to maintain firm A producing strictly positive output levels.

- In a parallel fashion we can determine firm B's profit maximizing production levels conditional on the expectation of firm A's behavior. If $q_A = 0$ then $\widetilde{q}_B = q_2$. Also, given the restriction $q_B \geq 0$, necessarily, $q_A < q_3$. As the expectation on firm A shifts from zero to q_3, the points where marginal revenue equates marginal cost will move along the line HI in figure 3.4, and the aggregate production level will increase from q_2 until q_3.

Combining these arguments, we see that for an equilibrium to exist both firms have to produce positive levels of output and the segments q_2q_3 and q_4q_6 have to intersect at least in one point. A priori there is no guarantee that such intersection will occur. The illustration provided in figure 3.4 such intersection does not exist because of the large difference between the marginal costs.

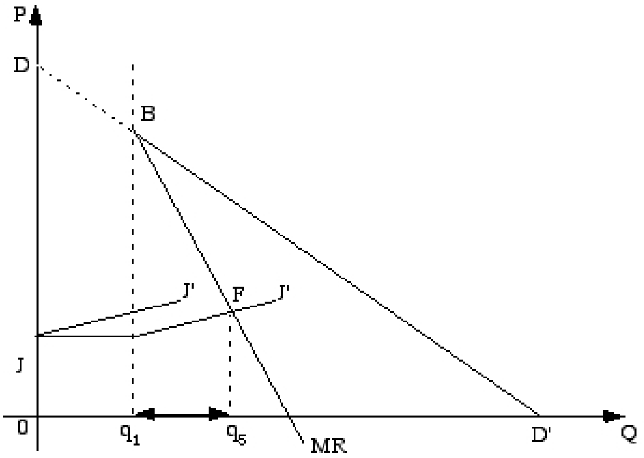

Figure 3.5: Existence of Cournot equilibrium (ii).

A formal approach.

Recall the assumptions introduced in section 3.1.1. The concavity of the profit function means $\frac{\partial^2 \Pi_i}{\partial q_i^2} \leq 0$. This ensures that there is always a production level $\widetilde{q}_i$ maximizing profits for any $\sum_{j \neq i} q_j$. Let us examine now the conditions under which the profit function is concave:

- a) marginal revenue is decreasing in q_i (i.e concave demand), and firm i's marginal cost is constant or increasing (convex cost). Formally, $MR_i^{'} < 0$ and $C_i^{''} \geq 0$.

- b) firm i's marginal revenue is increasing in q_i (i.e convex demand), and firm i's marginal cost grows at the same rate as or faster than the marginal revenue. Formally, $MR_i^{'} > 0$ i $C_i^{''} \geq MR_i^{'}$.

- c) firm i's marginal revenue is decreasing in q_i (i.e concave demand), and firm i's marginal cost grows at the same rate as or slower than the marginal revenue. Formally, $MR_i^{'} < 0$ i $|MR_i^{'}| \geq |C_i^{''}|$.

Summarizing,

$$\frac{\partial^2 \Pi_i(q)}{\partial q_i^2} = 2f'(\sum_i q_i) + q_i f^{''}(\sum_i q_i) - C_i^{''}(q_i)$$

$$\frac{\partial^2 \Pi_i(q)}{\partial q_i^2} < 0 \text{ if } \begin{cases} f^{''} < 0 \text{ and } C_i^{''} > 0 \\ f^{''} > 0, C_i^{''} > 0 \text{ and } |2f'| > q_i f^{''} - C_i^{''} \\ f^{''} < 0, C_i^{''} < 0 \text{ and } |2f' + q_i f^{''}| > |C_i^{''}| \end{cases}$$

Note that these assumptions are sufficient, but not necessary to guarantee the existence of a Cournot equilibrium. The interested reader will find more general approaches in e.g. Friedman (1977), Mas Colell et al. (1995), Okuguchi (1976), or Vives (1999).

Theorem 3.1. *Whenever assumptions **S1** to **S3** are fulfilled, there will be an interior Cournot equilibrium,*

Proof. We will divide the proof in three parts. First, we will show that assumptions **S1** to **S3** ensure well-defined reaction functions (lemma 3.1); next, we will show that these reaction functions are continuous (lemma 3.2); finally we will verify that we can apply Brower's fix point theorem.

Lemma 3.1. *Whenever assumptions **S1** to **S3** are fulfilled, there will be a well-defined reaction function for every firm.*

Proof. Let $\Im_{-i} \stackrel{\text{def}}{=} [0,\overline{Q}] \times [0,\overline{Q}] \times [0,\overline{Q}] \times \overset{\text{(n-1) times}}{\cdots} \times [0,\overline{Q}]$. Consider an arbitrary production plan $q_{-i} \in \Im_{-i}$. Given that $\pi_i(q_i, q_{-i})$ is continuous in $q_i,\ q_i \in [0,\overline{Q}]$ and strictly concave, and given that $[0,\overline{Q}]$ is compact we can write the first order condition of the profit maximization problem

$$\frac{\partial \pi_i(q_i, q_{-i})}{\partial q_i} = f(Q) + q_i f'(Q) - C_i^{'}(q_i) = 0,$$

as a function $q_i = w_i(q_{-i})$ called firm i's reaction function. It tells us firm i's profit maximizing strategy conditional to its expectation on the behavior of the $(n-1)$ rival firms. □

Let us now define a one-to-one continuous mapping, $w(q)$, of the compact set $\Im$ on itself,

$$w(q) = \Big(w_1(q_{-1}), w_2(q_{-2}), \dots, w_n(q_{-n})\Big)$$

Lemma 3.2. $w_i(q_{-i})$ *is a continuous function.*

Proof. Let $\{\overline{q}_{-i}\}_{\tau=1}^{\infty}$ a sequence of strategy vectors in $\Im_{-i}$, such that $\lim_{\tau\to\infty} q_{-i}^{\tau} = q_{-i}^{o}$.

Since $\Im_{-i}$ is compact, we know that $q_{-i}^{o} \in \Im_{-i}$.

The sequence $\{\overline{q}_{-i}\}_{\tau=1}^{\infty}$ allows us to obtain a sequence $\{w_i(q_{-i}^{\tau})\}_{\tau=1}^{\infty}$ where $w_i(q_{-i}^{\tau}) \in [0,\overline{Q}]$.

Let,

$$q_i^{\tau} = w_i(q_{-i}^{\tau})$$
$$q_i^{o} = w_i(q_{-i}^{o}).$$

We say that w_i is continuous if $\lim_{\tau\to\infty} q_i^{\tau} = q_i^{o}$.

By definition,

$$\pi_i\Big(w_i(q^{\tau}_{-i}), q^{\tau}_{-i}\Big) \geq \pi(q_i, q^{\tau}_{-i}),\ q_i \in [0, \overline{Q}].$$

Since the profit function is continuous,

$$\lim_{\tau\to\infty} \pi_i\Big(w_i(q^{\tau}_{-i}), q^{\tau}_{-i}\Big) = \pi_i\Big(\lim_{\tau\to\infty} w_i(q^{\tau}_{-i}), q^{o}_{-i}\Big)$$
$$\lim_{\tau\to\infty} \pi(q_i, q^{\tau}_{-i}) = \pi(q_i, q^{o}_{-i}),$$

so that we can write,

$$\pi_i\Big(\lim_{\tau\to\infty} w_i(q^{\tau}_{-i}), q^{o}_{-i}\Big) \geq \pi(q_i, q^{o}_{-i}),\ q_i \in [0, \overline{Q}].$$

Given that $w_i(q_{-i})$ is a single-valued function,

$$\lim_{\tau\to\infty} w_i(q^{\tau}_{-i}) = w_i(q^{o}_{-i})$$

or equivalently,

$$\lim_{\tau\to\infty} q^{\tau}_i = q^{o}_{-i},$$

so that $w_i(q_{-i})$ is a continuous function. □

We present now (without proof) Brower's fix point theorem[1].

Theorem 3.2 (Brower)**.** *Let X be a convex and compact set in* $\mathbf{R^n}$*. Let $f : X \to X$ be a continuous application associating a point $f(x)$ in X to each point x in X. Then there exists a fixed point $\widehat{x} = f(\widehat{X})$.*

Figure 3.6 illustrates the theorem where $X = [0, 1]$. In cases A and B we have two fixed points at $0, 1$. In case C there are three fixed points at $0, 1, \widehat{x}$. Finally, in case D there is a unique fix point at $\widehat{x}$.

We can apply Brower's fix point theorem, given that $\Im$ is compact and $w(q)$ is continuous. Therefore, we can guarantee that there is at least a point q^* such that $w(q^*) = q^*$, where q^* is the Cournot equilibrium. □

3.1.5 Uniqueness of the Cournot equilibrium.

An intuitive approach.

The analysis of existence of equilibrium identifies conditions guaranteeing the presence of equilibrium vectors $(q^*_1, q^*_2, \ldots, q^*_n)$. This potential multiplicity of

[1] On fix point theorems see Border (1992).

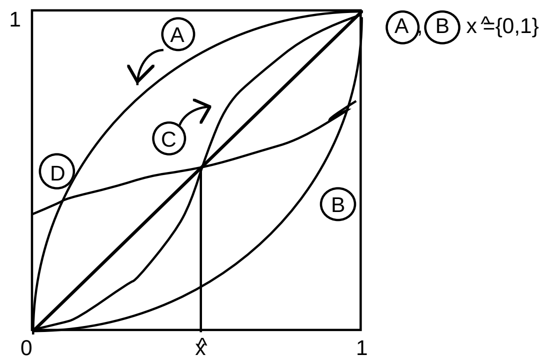

Figure 3.6: Brower's fixed point theorem.

equilibria may rise a problem. To illustrate, consider a Cournot duopoly with two equilibria (q_1^*, q_2^*) and (y_1^*, y_2^*). This means that say, firm 1 knows that if firm 2 chooses q_2^* its best reply is q_1^*, while if firm 2 chooses y_2^* its best reply is y_1^*; the problem is precisely that firm 1 does not know firm 2's decision. It only makes conjectures. It may well happen that firm 1 conjectures that firm 2 will choose q_2^* when it turns out that firm 2's choice is y_2^*. In this case we end up with a production plan (q_1^*, y_2^*) that generally will not be an equilibrium vector of strategies. In other words, there may appear a coordination problem. How can we be sure that firms will "point at the same equilibrium production plan"? A possible way out of this problem is to study the conditions under which there is a unique equilibrium. Before going into this, let us try to understand why there may appear a multiplicity of equilibria. Let us look at figure 3.7 where $D'D$ denotes the demand curve and FF', GG', JJ', KK' are some isoprofit curves of firm B.

- Assume that firm B conjectures that firm A will produce q_1 units. Its residual demand is CD'.

- Observe now point E. This is a tangency point between the demand curve CD' and the isoprofit curve FF'. This implies that at E firm B's profits are maximized. The associated output level at this point is $q_1q_2 = Oq_1$.

We can perform a parallel analysis for firm A. That is, given a conjecture q_1 on firm B, q_1q_2 is the production level maximizing firm A profits. Accordingly, E represents a Cournot equilibrium where the associated production plan is both firms producing Oq_1.

- We can repeat this argument for the point E', so that E' is also a Cournot equilibrium point where the associated production plan is both firms producing $Oq_3 = q_3q_4$.

- In this example the multiplicity of equilibria arises as a consequence of the lack of concavity of the demand function.

What additional assumption do we have to introduce to guarantee the unique-

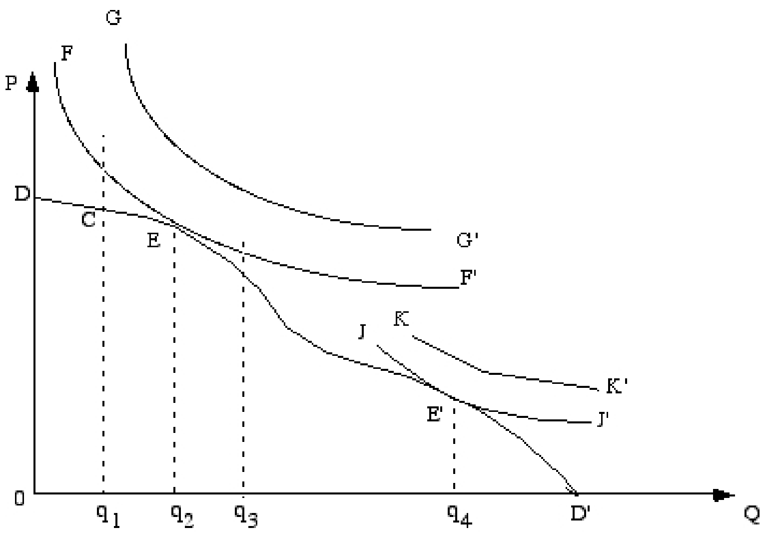

Figure 3.7: An instance of non-uniqueness of equilibrium

ness of the Cournot equilibrium?

A formal approach.

We need to restrict further some of the assumptions introduced in the analysis of existence. In particular, we maintain assumptions **S1, S2** and introduce the following assumption:

S4 The profit function $\pi_i(q)$ is continuous, twice continuously differentiable, and $\forall q,\ q \gg 0,\ Q < \overline{Q}$ satisfies,

$$\frac{\partial^2 \pi_i(q)}{\partial q_i^2} + \sum_{j \neq i} \left| \frac{\partial^2 \pi_i(q)}{\partial q_i \partial q_j} \right| < 0$$

Note first that **S4** implies that the set $\Im$ is compact. Clearly, **S4** is more restrictive in the sense that the class of functions that satisfies it is smaller. To see it, we can rewrite **S4** as

$$2f'(Q) + q_i f''(Q) - C_i''(q_i) + (n-1)|f'(Q) + q_i f''(Q)| < 0 \qquad (3.2)$$

Consider the case $f'' < 0$, so that $|f'(Q) + q_i f''(Q)| = -\big(f'(Q) + q_i f''(Q)\big)$. Thus, (3.2) can be rewritten as

$$-(n-3)f'(Q) - (n-2)q_i f''(Q) - C_i''(q_i) < 0. \qquad (3.3)$$

For $n > 3$ the first two terms are positive. This means that increasing values of n require increasing values of $C_i''(q_i)$ to verify (3.3).

Summarizing, assumptions **S1, S2, S4** guarantee that $w(q)$ is a contraction.

Definition 3.9. *Consider two vectors $q^{'}_{-i}$ and $q^{''}_{-i}, i = 1, 2, \ldots, n$. It is said that $w(q)$ is a contraction if*

$$\left| w_i(q^{'}_{-i}) - w_i(q^{''}_{-i}) \right| < \| q^{'}_{-i} - q^{''}_{-i} \|$$

That is, when all the competitor firms vary their strategies in a certain amount, firm i's best reply varies in a smaller amount.

We will now introduce a theorem without proof:

Theorem 3.3. *Let $f : \mathbf{R}^l \to \mathbf{R}^l$ be a contraction. Then, f has a unique fixed point.*

We can use this theorem to obtain the result we are after:

Theorem 3.4. *Assume **S1, S2, S4**. Then $w(q)$ is a contraction and q^* is the unique Cournot equilibrium.*

3.1.6 An alternative approach to the existence and uniqueness of Cournot equilibrium.

Szidarovsky and Yakowitz (1977) propose a different approach to prove the existence and uniqueness of equilibrium in a Cournot model. This approach relates the individual profit maximizing level of output with the aggregate production in the industry and shows that this relation is monotone decreasing.

Let q_i denote firm i's production level, $i = 1, 2, \ldots, n$, let $Q \equiv \sum_i q_i$denote the industry aggregate output, let $p = f(Q)$ be the inverse demand function, and $C_i(q_i)$ firm i's cost function.

Assumption 3.1. *The inverse demand function is continuous, differentiable, concave and cuts the horizontal axis. Formally, $f'(Q) < 0,\ f''(Q) \leq 0$, for $Q \in [0, \overline{Q}]$ where $\overline{Q}$ is such that $f(\overline{Q}) = 0$.*

Assumption 3.2. *The cost function is continuous, differentiable, and convex. Formally, $C^{'}_i(q_i) > 0,\ C^{''}_i(q_i) \geq 0$, for $q_i \in [0, \overline{Q}]$.*

Assumption 3.3. $\forall i,\ f(0) > C^{'}_i(0)$.

Assumption 3.3 is technical. It says that every firm is willing to produce at least a small quantity if it would be a monopolist. This is so because when $Q = 0$ (and thus $q_i = 0$, marginal revenue ($f(0)$) is above marginal cost $C^{'}_i(0)$, so that it is profitable for the firm to produce an arbitrarily small output.

Friedman (1977) shows that when assumption 3.3 is verified for at least one firm, then $Q^* > 0$. He also shows that if Q^* is a Cournot equilibrium, then it is not on the frontier of the space of outcomes, $\Pi(\Im)$ (i.e. is not Pareto optimal).

Let us now define firm i's optimal output consistent with Q as

$$q_i(Q) = \begin{cases} q & \text{where } \ q \geq 0 \text{ and } f(Q) = C_i^{'}(q) - qf'(Q), \\ 0 & \text{otherwise.} \end{cases}$$

That is, firm i's strategy is to produce a non negative output only when it allows to maximize profits. Otherwise, the firm does not enter the market (or is not active in the market).

Lemma 3.3. *$q_i(Q)$, when positive, is continuous and monotone decreasing in Q.*

Proof. The continuity comes from the continuity of the inverse demand and cost functions.

$q_i(Q)$ is decreasing because

$$\frac{\partial q_i(Q)}{\partial Q} = \frac{r_i^{'}}{1 + r_i^{'}}$$

where $r_i^{'} \in [-1, 0]$ denotes firm i's reaction function. It is decreasing from the concavity of the profit function. □

- Consider $Q = 0$. Assumption 3.3 tells us that $\sum_i q_i(0) \geq 0$;
- Consider $Q = \overline{Q}$. From the definition of $q_i(Q)$, q has to verify $C_i^{'}(q) = qf'(\overline{Q})$. Hence, $q = 0$ and $\sum_i q_i(\overline{Q}) = 0$;

Lemma 3.3 says that $q_i(Q)$ is continuous and decreasing for any Q giving rise to $q_i(Q) > 0$. Accordingly, $\sum_{i=1}^{n} q_i(Q)$ is also continuous and decreasing. Hence, there can only exist one value Q^* for which $\sum_{i=1}^{n} q_i(Q^*) = Q^*$. We conclude that Q^* is the only Cournot equilibrium and $\Big(q_1(Q^*), q_2(Q^*), \ldots, q_n(Q^*)\Big)$ its associated production plan.

An alternative way to construct the argument is provided by Tirole (1988, pp. 224-225). Assumptions 3.1 and 3.2 guarantee that the profit function is concave. Therefore, we can obtain the reaction functions $r_i(q_{-i})$. To guarantee that the reaction functions will intersect we need to assume 3.3 and also

Assumption 3.4. $r_j^{-1}(0) > r_i(0) = q_i^m$.

This says that firm i's output inducing firm j to remain inactive exceeds firm i's monopoly output. In other words, the intercept of firm j reaction function on the axis measuring firm i's output, is above firm i's monopoly output.

To illustrate the argument let us consider the data in problem 7. The system of first order conditions is,

$$840 - 2q_1 - q_2 = 0,$$
$$900 - q_1 - 2q_2 = 0.$$

We can rewrite it as,

$$q_1(Q) = 840 - Q, \tag{3.4}$$
$$q_2(Q) = 900 - Q. \tag{3.5}$$

Adding up (3.4) i (3.5) we obtain,

$$q_1(Q) + q_2(Q) = 1740 - 2Q. \tag{3.6}$$

In equilibrium, $q_1(Q) + q_2(Q)$ on the left hand side of (3.6) must be equal to Q on the right hand side. Thus, solving (3.6) for this common value we obtain,

$$Q = 580.$$

Substituting this value of Q in (3.4) and (3.5) we obtain the equilibrium production plan $q_1 = 260, \ q_2 = 320$.

Note that in equation (3.6), $q_1 + q_2$ decreases as Q increases; therefore, given this monotonic decreasing relation there can only be a value of Q satisfying $Q = 1740 - 2Q$.

More general analyses in this line are Koldstad and Mathiesen (1987), Gaudet and Salant (1991), Van Long and Soubeyran (2000) and Watts (1996).

3.1.7 Strategic complements and substitutes.

We have seen that the set of reaction functions characterizes the Cournot equilibrium. We want to study some properties of these reaction functions with some more detail. Essentially, reaction functions show the strategic dependence among firms. The nature of this dependence is crucial in the determination of the properties of oligopoly models. For ease of exposition, let us consider a duopolistic market.

We obtain say firm 1's reaction function from its profit maximization problem, given its expectation on the decision of the rival, q_2:

$$\frac{\partial \pi_1(q_1, q_2)}{\partial q_1} = f(Q) + q_1 \frac{df}{dQ} - \frac{dC_1(q_1)}{q_1} = 0. \tag{3.7}$$

The strategic nature of the relation between firms is given by the slope of the reaction function. The slope is obtained by differentiating (3.7):

$$\frac{\partial^2 \pi_1(q_1,q_2)}{\partial q_1^2}\frac{dq_1}{dq_2}\Big|_{foc} + \frac{\partial^2 \pi_1(q_1,q_2)}{\partial q_1 \partial q_2} = 0, \text{ that is}$$

$$\frac{dq_1}{dq_2}\Big|_{foc} = -\frac{\frac{\partial^2 \pi_1(q_1,q_2)}{\partial q_1 \partial q_2}}{\frac{\partial^2 \pi_1(q_1,q_2)}{\partial q_1^2}}. \tag{3.8}$$

Since $-\dfrac{\partial^2 \pi_1(q_1,q_2)}{\partial q_1^2} > 0$, from the second order condition, it turns out that the slope of the reaction function is given by the sign of the numerator in (3.8). Let us then look at that numerator,

$$\frac{\partial^2 \pi_1(q_1,q_2)}{\partial q_1 \partial q_2} = \frac{\partial}{\partial q_2}\Big[\frac{\partial \pi_1(q_1,q_2)}{\partial q_1}\Big] = \frac{df}{dQ} + q_1\frac{d^2 f}{dQ^2}. \tag{3.9}$$

Therefore, when demand is concave, the crossed partial derivative is negative and so is the slope of the reaction function. If demand is strictly convex, the last term on the right hand side of (3.9) is positive and the reaction function may be positively sloped. Following Bulow, Geanakoplos and Klemperer (1985), we say that the actions of the two firms are *strategic complements* if $\dfrac{\partial^2 \pi_1(q_1,q_2)}{\partial q_1 \partial q_2} > 0$ and *strategic substitutes* if $\dfrac{\partial^2 \pi_1(q_1,q_2)}{\partial q_1 \partial q_2} < 0$. We will see that prices are often strategic complements while quantities are often strategic substitutes[2]. Nevertheless, the nature of the strategic relations among competitors has to be examined case by case. A throughout investigation of games with strategic complementarities can be found in Amir (1996, 2005) and Vives (1999, 2005a, 2005b).

3.1.8 Cournot and conjectural variations.

Consider a homogenous market with inverse demand function $p = f(Q)$. There are n firms, each with a technology $C_i(q_i),\ i = 1, 2, \dots, n$. Accordingly, firm i's profit function is given by $\Pi_i(q) = f(Q)q_i - C_i(q_i)$. Assuming that demand and cost functions satisfy the sufficient conditions for the existence of a unique

[2]Martin (2002, pp. 21-27) show some examples where that relation does not hold.

equilibrium, this is characterized by the system of first order conditions,

$$\frac{\partial \Pi_i(q)}{\partial q_i} = f(Q) + q_i f'(Q)\frac{\partial Q}{\partial q_i} - C_i'(q_i) = 0, \ \forall i$$

where

$$\frac{\partial Q}{\partial q_i} = \frac{\partial \sum_{j=1}^{n} q_j}{\partial q_i} = 1 + \frac{\partial \sum_{j \neq i} q_j}{\partial q_i}.$$

The term $\frac{\partial \sum_{j \neq i} q_j}{\partial q_i}$ captures the strategic interdependence among firms and is called the *conjectural variation*[3]. It shows how firm i forms its expectations on the behavior of its rivals. In Cournot every firm expects that its rivals will not change their decisions when it varies its production level infinitesimally, that is $\dfrac{\partial \sum_{j \neq i} q_j}{\partial q_i} = 0.$

3.1.9 The geometry of the Cournot model.

To proceed with the graphical representation, we will assume a duopolistic industry. We will also assume linear demand and costs for simplicity, $p = a - b(q_1 + q_2)$ and $C_i(q_i) = c_0 + cq_i, \ i = 1, 2.$

Isoprofit curves.

An isoprofit curve is the locus of points (q_1, q_2) associated to the same level of profits. Consider firm i. Its profit function is,

$$\Pi_i(q) = q_i(a - b(q_i + q_j)) - c_0 - c_i q_i$$

Fix a level of profits $\Pi_i(q) = \overline{\Pi}$, so that

$$\overline{\Pi} = q_i(a - b(q_i + q_j)) - c_0 - c_i q_i, \quad \text{that is}$$
$$q_j = -\frac{1}{b}\Big(\frac{\overline{\Pi} + c_0}{q_i} + c - a\Big) - q_i$$

This is the expression of a representative isoprofit curve for firm i. Given an expectation on the output level of firm j, the isoprofit equation gives all firm i's production levels consistent with a profit level $\overline{\Pi}$.

Let us study the shape of this function.

[3]On conjectural variations see section 3.9

- the slope of the isoprofit curve is given by:

$$\frac{\partial q_j}{\partial q_i}\bigg|_{\overline{\Pi}} = \frac{1}{b}\Big(\frac{\overline{\Pi} + c_0}{q_i^2}\Big) - 1.$$

- Accordingly, it has a critical point at:

$$q_i = \Big(\frac{\overline{\Pi} + c_0}{b}\Big)^{\frac{1}{2}}. \tag{3.10}$$

Note that this critical point is increasing in $\overline{\Pi}$:

$$\frac{\partial q_i}{\partial \overline{\Pi}} = \frac{1}{2b}\Big(\frac{\overline{\Pi} + c_0}{b}\Big)^{\frac{-1}{2}} > 0. \tag{3.11}$$

- Also, the isoprofit curve is strictly concave in the space (q_i, q_j):

$$\frac{\partial^2 q_j}{\partial q_i^2} = -\frac{2(\overline{\Pi} + c_0)}{b q_i^3} < 0.$$

Finally, we want to identify the extreme curves of the family of isoprofit curves.

- It should be clear that firm i will achieve the maximum level of profits as a monopoly, Π_i^m. The corresponding isoprofit curve will have only one feasible production plan $(q_i^m, 0)$, where $q_i^m = \frac{a-c}{2b}$. The level of profits is $\Pi_i^m = \frac{(a-c)^2}{4b} - c_0$. Hence, the isoprofit curve associated to this level of profits Π_i^m will be tangent to the q_i axis from below.

- The other extreme is associated with the situation of minimum profits for firm i. This appears when the production plan $(0, \widetilde{q}_j)$ maximizes firm i's profits. In other words, we are looking for the value $\widetilde{q}_j$ such that $\frac{\partial \Pi_i}{\partial q_i} = 0$ when $q_i = 0$. This is $\widetilde{q}_j = \frac{a-c}{b}$.

- We also want to characterize the function linking the maximum points of all the isoprofit family of curves. Given the linearity of demand and cost functions, it easy to see that it will also be a linear function. Consider an arbitrary isoprofit curve $\overline{\Pi}$. Its maximum with respect to q_i is given by (3.10). Substituting it in the equation of the corresponding isoprofit curve we obtain the associated value q_j. This is,

$$q_j = \frac{a-c}{b} - 2\big(\frac{\overline{\Pi} + c_0}{b}\big)^{\frac{1}{2}}. \tag{3.12}$$

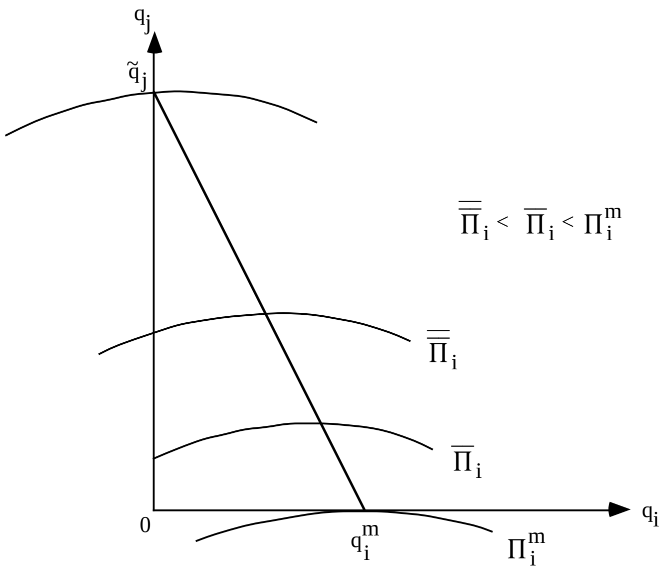

Figure 3.8: Firm i's isoprofit curves.

Compute now,

$$\frac{\partial q_j}{\partial \overline{\overline{\Pi}}} = \frac{-1}{b}\Big(\frac{\overline{\Pi} + c_0}{b}\Big)^{\frac{-1}{2}}. \tag{3.13}$$

Comparing (3.11) and (3.13) we see,

$$\frac{\partial q_i}{\partial \overline{\overline{\Pi}}} = -\frac{1}{2}\frac{\partial q_j}{\partial \overline{\overline{\Pi}}}.$$

That is, when there is a variation in the level of profits, the effect on q_i is half the effect on q_j regardless of the actual value of profits. This implies a linear relation between the set of maximum points of the family of isoprofit curves. This linear function has slope $-\frac{1}{2}$. Finally, to identify the expression of the linear function we substitute (3.10) in (3.12) to obtain,

$$q_i = \frac{a-c}{2b} - \frac{1}{2}q_j. \tag{3.14}$$

Figure 3.8 summarizes this discussion.

In a parallel fashion we can obtain firm j's family isoprofit curves (see figure 3.9). The linear function linking the set of maximum points is $q_j = \dfrac{a-c}{2b} - \dfrac{1}{2}q_i$.

Putting together the maps of isoprofit curves of both firms, and given the strict concavity of all of them, allows us to identify a locus of tangency points between

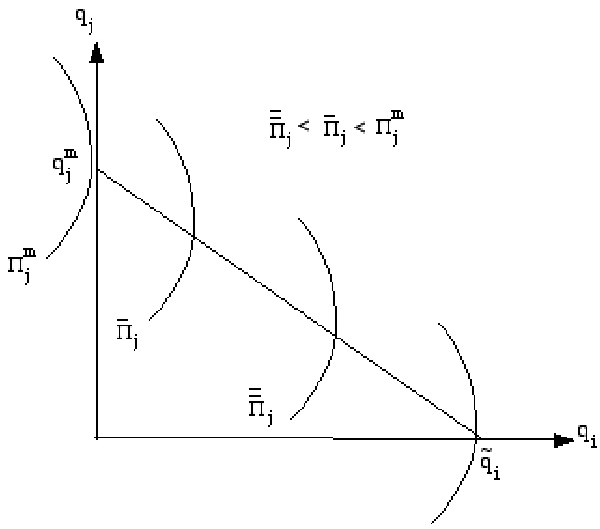

Figure 3.9: Firm j's isoprofit curves.

firm i's isoprofit curves and the corresponding of firm j as shown in figure 3.10. Note that each tangency point gives rise to a distribution of profits such that any alternative share of profits cannot make both firms better off simultaneously. In other words, the locus of tangency points is precisely the set of Pareto optimal production plans. Thus we can identify the associated distributions of profits. This Pareto optimal distribution of profits, are such that the sum of profits is maximum. That is, a tangency point between two isoprofit curves represents a production plan that maximizes the joint profits of the firms. The set of such production plans is thus a function whose extreme points in the space (q_i, q_j) are the monopoly outputs for every firm $(q_i^m, 0), (0, q_j^m)$.

Formally, we want to solve the following problem,

$$\max_{q_i, q_j}(\Pi_i + \Pi_j) = Q(a + bQ) - 2c_0 - cQ.$$

Its solution is a linear function $q_i = \dfrac{a-c}{2b} - q_j$, where the extremes correspond to the monopoly outputs and the slope is -1 as figure 3.10 illustrates.

Reaction functions.

A reaction function for a firm is the locus of profit maximizing production plans conditional to the expectation on the behavior of the rival firms. Formally, we obtain firm i's reaction function from the first order conditions of its profit maxi-

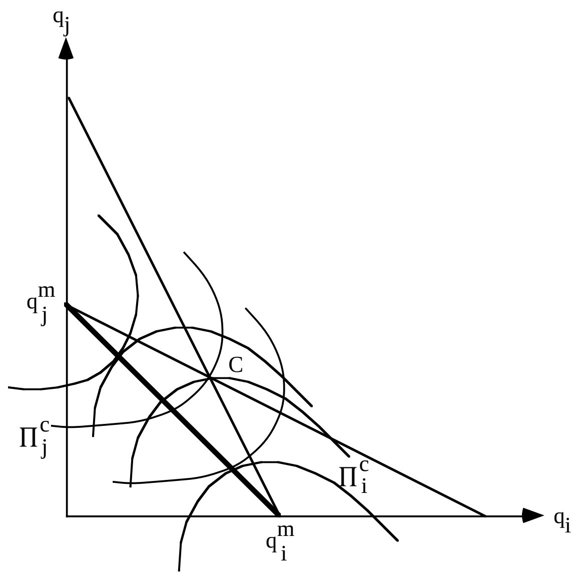

Figure 3.10: The loci of Pareto optimal points.

mization program:

$$\begin{aligned} \Pi_i(q) &= (a - bQ)q_i - c_0 - cq_i, \\ \frac{\partial \Pi_i(q)}{\partial q_i} &= a - c - bq_j - 2bq_i = 0, \\ q_i &= \frac{a-c}{2b} - \frac{1}{2}q_j. \end{aligned}$$

Note that this expression is the same as (3.14). This means that firm i's reaction function is precisely the function linking the maximum points of its family of isoprofit curves.

This relation between isoprofit curves and the reaction function should not be surprising. On the one hand, we are identifying firm i's best reply to any given expectation on the behavior of the rivals (reaction function); on the other hand, the maximum of an isoprofit curve tells us, for each profit level the production plan associated. Therefore, we are looking at the same problem from two different perspectives. Either from the point of view of production plans, or from the point of view of profit distributions.

From the definition of Cournot equilibrium, we can characterize it from the system of first order conditions. That is, a Cournot equilibrium is a production plan satisfying all firms reaction functions simultaneously. In terms of our linear example, the unique Cournot equilibrium is the production plan $(q_i^c, q_j^c) = (\frac{a-c}{3b}, \frac{a-c}{3b})$. In figure 3.10 this corresponds to the intersection of the reaction functions. It is worth noting that at this point the corresponding isoprofit curves are not tangent. Hence, as we have already seen, the Cournot equilibrium does not yield a Pareto optimal distribution of profits.

3.1.10 Cournot and the competitive equilibrium.

So far we have assumed that the number of firms in the industry, n, is given. If we relax this assumption two questions arise:

Quasi-competitiveness of Cournot equilibrium Does industry output increases as n increases?

Convergence to competitive equilibrium Does the Cournot equilibrium converge towards the competitive equilibrium as n increases?

These questions are interesting on two grounds.

On the one hand, we would like to have an approximation of the effect of oligopolistic markets on welfare. Even though a full answer to this question would demand a general equilibrium model, a partial equilibrium model will provide some intuition.

On the other hand, if it would turn out that the Cournot equilibrium is not sensibly different from a competitive equilibrium, oligopoly theory would be basically empty of any interesting (relevant) question.

Two illustrations.

The first illustration (see Martin (2002, pp. 18-19)) considers a duopoly with identical firms, that is both with the same technology of constant marginal cost, normalized to zero without loss of generality. Let Q^d denote the Cournot equilibrium output of this duopoly. Given the symmetry of the firms, such equilibrium output will be evenly split between both firms, i.e. $q_i^d = Q^d/2$. Accordingly, we can write the first order condition of the maximization problem of one firm as

$$f(Q^d) + \frac{Q^d}{2} f'(Q) \equiv 0, \text{ or}$$
$$2f(Q^d) = -Q^d f'(Q)$$

In general, with n symmetric firms we obtain

$$nf(Q) = -Qf'(Q).$$

In other words, the Cournot equilibrium price with n identical firms is characterized by the intersection of two curves $nf(Q)$ and $-Qf'(Q)$. The former is a

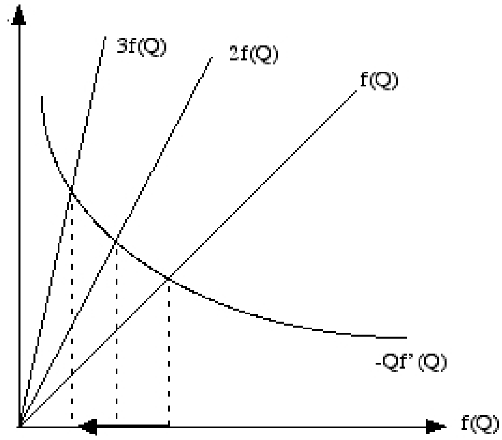

Figure 3.11: Cournot with increasing number of firms.

linear function, upward sloping and steeper as n increases. The latter usually will be decreasing. Figure 3.11 illustrates how the price falls as the number of firms increase. In the limit, as $n \to \infty$, the price will converge to zero (the marginal cost). That is, towards the long-run price of a perfectly competitive market.

To introduce the second illustration, let us recall that in a competitive market, the equilibrium is characterized by equating price and marginal cost. In oligopoly, we have already seen that in equilibrium marginal revenue equals marginal cost, so that firms obtain positive profits given the margin of the price over the marginal cost. To study "how far" is the Cournot equilibrium from the competitive equilibrium, we will assume that all firms behave competitively (i.e. as if they would be myopic enough not to realize the strategic interaction among them) so that they adjust their production levels to the point where price equals marginal cost. Then we will compare the resulting outcome with the Cournot outcome. This equilibrium concept, called *efficient point*, was first introduced by Shubik (1959).

Definition 3.10. *The "efficient point" is a production plan resulting from equating price to marginal cost for all firms simultaneously, i.e. we say that* $q^e \in \mathbf{R}^\mathbf{n}$ *is an efficient point if it solves the equation* $f(Q) = C_i^{'}(q_i),\ \forall i$.

To illustrate, consider an industry where n firms produce a homogeneous product all using the same constant marginal cost technology, and let market demand be linear.

$$C_i(q_i) = cq_i,\ c > 0,\ i = 1, 2, \ldots, n.$$
$$f(Q) = a - bQ,\ a, b > 0,\ Q = \sum_{i=1}^{n} q_i.$$

Cournot equilibrium is characterized by the solution of the system of first order conditions,

$$\frac{\partial \Pi(q)}{\partial q_i} = a - b(Q + q_i) - c = 0,\ i = 1, 2, \ldots, n. \tag{3.15}$$

By symmetry, we know that in equilibrium all firms will produce the same volume of output $q_i^c = q_j^c,\ \forall i, j; i \neq j$. Hence, we can rewrite 3.15 as,

$$a - c - b(n + 1)q_i^c = 0.$$

Accordingly,

$$q_i^c = \frac{a - c}{b(n + 1)},$$
$$Q^c = \frac{n(a - c)}{b(n + 1)},$$
$$\Pi_i(q^c) = \frac{(a - c)^2}{b(n + 1)^2}.$$

The efficient point equilibrium is characterized by the solution of $f(Q) = C_i'$. That is,

$$a - bQ = c, \quad \text{or} \quad Q^e = \frac{a - c}{b}. \quad \text{Thus} \quad q_i^e = \frac{a - c}{nb}.$$

Comparing both equilibria we should note that,

$$q_i^c \leq q_i^e,\ Q^c \leq Q^e,\ P^c \geq P^e. \text{ Also,}$$
$$\lim_{n \to \infty} Q^c = \frac{a - c}{b} = Q^e.$$

Does this mean that we can assume competitive behavior on oligopolistic firms without losing much in the understanding of the behavior of the market? The answer is NO. Let us modify slightly the example above introducing fixed costs so that,

$$C_i(q_i) = k + cq_i,\ k, c > 0$$

The production plan characterizing the Cournot equilibrium is the same as before, but the associated level of profits is now,

$$\Pi_i(q^c) = \frac{(a-c)^2}{b(n+1)^2} - k.$$

Naturally, firms in the market must obtain non-negative profits. Accordingly, in this scenario there is an upper bound on the number of firms that can be active in the market,

$$\Pi_i(q^c) \geq 0 \ \Rightarrow \frac{(a-c)^2}{b(n+1)^2} - k \geq 0 \ \Rightarrow n \leq \frac{a-c}{\sqrt{bk}} - 1$$

The important feature is that there is a *finite* number of firms in equilibrium. This implies that now it does not make sense to consider the limit behavior as $n \to \infty$ and, in turn, it would be incorrect to identify the competitive and oligopolistic behavior. Also, note (a) $\frac{\partial n}{\partial k} < 0$, so that the higher the fixed cost the smaller the number of firms active in the market, and (b) $\frac{\partial \Pi_i}{\partial n} < 0$, Cournot profits are decreasing in n.

When there are no fixed costs, the producer surplus coincides with the profit of the firm. We know that the rule price = marginal cost yields zero profit to the firm and characterizes a Pareto optimal equilibrium. Thus, it should not be surprising that $n \to \infty$ approaches the oligopolistic behavior to a Pareto optimal solution.

With fixed costs, the rule price= marginal cost implies negative profits. In other words, the presence of fixed costs limits the possibility of entry in the market. There will be a critical number n^* of incumbents. Further entry will imply that the potential profits cannot cover the fixed cost.

Quasi-Competitiveness of Cournot equilibrium [Comparative statics].

A conjecture often found in models of Industrial Organization is that in equilibrium firms' profits are decreasing in the number of competitors. We will examine now the conditions to guarantee this conjecture. The quasi-competitiveness of the equilibrium is a milder approach to the convergence to the competitive equilibrium. We have seen above that with a finite feasible number of firms in the market we cannot address the question of convergence. Nevertheless, we can still study some comparative statics and obtain intuition on whether as far as possible in terms of the number of firms, the oligopolistic behavior approaches the competitive market.

Let us follow Telser (1988) and consider an industry with n firms described by the following assumptions[4]:

[4] A more general analysis is found in Amir and Lambson (1996).

Assumption 3.5. *All firms use the same technology,*

$$C_i(q_i) = cq_i,\ i = 1, 2, \dots, n$$

Assumption 3.6. *All firms produce a homogeneous good. Market demand* $p = f(Q),\ Q = \sum_{i=1}^{s} q_i$, *is downward sloping and differentiable.*

Thus, the profit function of a firm is given by $\Pi(q) = (p - c)q_i$.

The Cournot equilibrium is a production plan $(q_1^0, q_2^0, \dots, q_i^0, \dots, q_n^0)$ characterized by the solution of the system of n first order conditions,

$$\frac{\partial \Pi(q)}{\partial q_i} = f(Q) - c + q_i \frac{\partial f}{\partial Q} = 0,\ i = 1, 2, \dots, n \tag{3.16}$$

By symmetry, we know that in equilibrium all firms will produce the same volume of output,

$$q_1^0 = q_2^0 = \dots = q_i^0 = \dots = q_n^0 = q^*,$$

so that,

$$Q = nq^*. \tag{3.17}$$

How does Π_i vary with n? To answer this question let us consider[5],

$$\begin{aligned}\Pi^* &= (f(Q) - c)q^* \\ \frac{\partial \Pi^*}{\partial n} &= (f(Q) - c)\frac{\partial q^*}{\partial n} + q^* \frac{\partial f}{\partial Q}\frac{\partial Q}{\partial n}.\end{aligned}$$

From (3.16), we know

$$f(Q) - c = -q_i \frac{\partial f}{\partial Q} = -q^* \frac{\partial f}{\partial Q},$$

so that,

$$\begin{aligned}\frac{\partial \Pi^*}{\partial n} &= -q^* \frac{\partial f}{\partial Q}\frac{\partial q^*}{\partial n} + q^* \frac{\partial f}{\partial Q}\frac{\partial Q}{\partial n} \\ &= q^* \frac{\partial f}{\partial Q}\left(\frac{\partial Q}{\partial n} - \frac{\partial q^*}{\partial n}\right).\end{aligned}$$

Given that $\dfrac{\partial f}{\partial Q} < 0$,

$$sgn \frac{\partial \Pi^*}{\partial n} = -sgn\left(\frac{\partial Q}{\partial n} - \frac{\partial q^*}{\partial n}\right).$$

[5]Given the symmetry of the model we can neglect the subindex i.

Hence, the answer to our question depends on the effect of n on both Q and q^*.

Differentiating (3.16) with respect to n we obtain,

$$\begin{aligned}&\frac{\partial f}{\partial Q}\frac{\partial Q}{\partial n}+\frac{\partial q^*}{\partial n}\frac{\partial f}{\partial Q}+q^*\frac{\partial^2 f}{\partial Q^2}\frac{\partial Q}{\partial n}=\\&\frac{\partial Q}{\partial n}\left(\frac{\partial f}{\partial Q}+q^*\frac{\partial^2 f}{\partial Q^2}\right)+\frac{\partial q^*}{\partial n}\frac{\partial f}{\partial Q}=0.\end{aligned} \tag{3.18}$$

Differentiating (3.17) with respect to n we obtain,

$$\begin{aligned}&\frac{\partial Q}{\partial n}=q^*+n\frac{\partial q^*}{\partial n} \text{ or,}\\&\frac{\partial Q}{\partial n}-n\frac{\partial q^*}{\partial n}=q^*.\end{aligned} \tag{3.19}$$

Solving (3.18) and (3.19) for $\frac{\partial Q}{\partial n}$ and $\frac{\partial q^*}{\partial n}$, we obtain,

$$\frac{\partial q^*}{\partial n}=-\frac{q^*\left(\frac{\partial f}{\partial Q}+q^*\frac{\partial^2 f}{\partial Q^2}\right)}{\frac{\partial f}{\partial Q}(n+1)+Q\frac{\partial^2 f}{\partial Q^2}}, \tag{3.20}$$

$$\frac{\partial Q}{\partial n}=\frac{q^*\frac{\partial f}{\partial Q}}{\frac{\partial f}{\partial Q}(n+1)+Q\frac{\partial^2 f}{\partial Q^2}}, \tag{3.21}$$

$$\frac{\partial Q}{\partial n}-\frac{\partial q^*}{\partial n}=\frac{q^*\left(2\frac{\partial f}{\partial Q}+q^*\frac{\partial^2 f}{\partial Q^2}\right)}{\frac{\partial f}{\partial Q}(n+1)+Q\frac{\partial^2 f}{\partial Q^2}}. \tag{3.22}$$

Given the assumptions 3.5 and 3.6 we cannot sign equations (3.20), (3.21), (3.22). Therefore we need some additional restrictions on the demand function.

Assumption 3.7. *Market demand is a concave function,*

$$\frac{\partial^2 f}{\partial Q^2}\leq 0.$$

Using assumption 3.7 in (3.20) and (3.21) we get,

$$\begin{aligned}&\frac{\partial q^*}{\partial n}<0,\\&\frac{\partial Q}{\partial n}>0. \text{ Therefore,}\\&\frac{\partial Q}{\partial n}-\frac{\partial q^*}{\partial n}>0 \quad \text{and} \quad \frac{\partial \Pi}{\partial n}<0.\end{aligned}$$

Alternatively, we could have introduced an assumption on the marginal revenue function. Recall that it is given by,

$$\frac{\partial f(Q)q_i}{\partial q_i} \equiv MR = f + q^* \frac{\partial f}{\partial Q}.$$

Assume now,

Assumption 3.8.

$$\frac{\partial MR}{\partial Q} = \frac{\partial f}{\partial Q} + q^* \frac{\partial^2 f}{\partial Q^2} < 0.$$

Note that assumption 3.8 is milder that assumption 3.7. In particular it admits some convex demand functions.

Using the alternative assumption 3.8 in (3.20) and (3.21) we obtain,

$$\frac{\partial q^*}{\partial n} = -\frac{q^* \dfrac{\partial MR}{\partial Q}}{n\dfrac{\partial MR}{\partial Q} + \dfrac{\partial f}{\partial Q}} < 0,$$

$$\frac{\partial Q}{\partial n} = \frac{q^* \dfrac{\partial f}{\partial Q}}{n\dfrac{\partial MR}{\partial Q} + \dfrac{\partial f}{\partial Q}} > 0, \quad \text{so} \quad \text{that}$$

$$\frac{\partial Q}{\partial n} - \frac{\partial q^*}{\partial n} > 0 \Longrightarrow \frac{\partial \Pi}{\partial n} < 0.$$

We have thus proved the following,

Lemma 3.4. *Consider an industry with n firms described by assumptions 3.5, 3.6 and 3.7 (or 3.8). Then, the Cournot equilibrium is quasi-competitive.*

The next question is whether we can extend the lemma to an arbitrarily large number of firms. In the limit this means that $n \approx n + 1$. Substituting in (3.20), (3.21), and (3.22) we obtain the same conclusion. To summarize,

Proposition 3.2. *(a) When the number of firms in the industry n is finite, the Cournot equilibrium is quasi-competitive if assumptions 3.5, 3.6 and 3.7 (or 3.8) hold.*

(b) If n is arbitrarily large, assumptions 3.5, 3.6 and 3.8 guarantee the quasi-competitiveness of the Cournot equilibrium.

In other words, in equilibrium the aggregate output of the industry is increasing in n, and the firms' output and profits and the market price are decreasing in n:

$$\frac{\partial Q^*}{\partial n} > 0, \quad \frac{\partial q^*}{\partial n} < 0, \quad \frac{\partial \Pi^*}{\partial n} < 0, \quad \frac{\partial f(Q^*)}{\partial n} < 0.$$

Convergence of Cournot equilibrium towards the competitive equilibrium.

To obtain convergence of the of Cournot equilibrium towards the competitive equilibrium we need to introduce the following assumptions,

Assumption 3.9.

$$\exists G \in (0, \infty) \textit{ s.t. } \left| f'(Q) \right| \leq G.$$

Assumption 3.10.

$$\exists \overline{Q} \in (0, \infty) \textit{ s.t. } f(Q) = 0 \ \forall Q \equiv \sum_i q_i \geq \overline{Q}.$$

That is the demand function eventually cuts the price axis, and also cuts the horizontal axis at $\overline{Q}$.

Assumption 3.11.

$$\begin{aligned} &C_i(q_i) \textit{is differentiable}, \ q_i \in [0, \overline{Q}], \\ &C_i(0) = 0, \\ &C_i^{'}(q_i) > 0, \\ &C_i(q_i) = C_j(q_j), \ \forall i, j; \ i \neq j; \ i, j = 1, 2, \ldots, n. \end{aligned}$$

These assumptions, as we know, guarantee that the equilibrium is symmetric and interior,

$$Q^* \equiv \sum_i q_i^* = nq_i^* < \overline{Q},$$

so that, $q_i^* < \dfrac{\overline{Q}}{n}$. Hence,

$$\lim_{n \to \infty} q_i^* = 0, \ \forall i. \tag{3.23}$$

Also, (3.23) implies,

$$\lim_{n \to \infty} q_i^* f'(Q^*) = 0, \ \forall i. \tag{3.24}$$

Firm i's profit function is,

$$\Pi_i(q) = q_i f(Q) - C_i(q_i),$$

and the first order condition of the profit maximization program is,

$$f(Q) + q_i f'(Q) - C_i^{'}(q_i) = 0.$$

Evaluating it at the equilibrium we obtain,

$$f(Q^*) + q_i^* f'(Q^*) - C_i^{'}(q_i^*) \equiv 0. \tag{3.25}$$

Expressions (3.24) and (3.25) together imply that when $n \to \infty$, then $f(Q^*) = C_i'(q_i^*)$. That is, price equals marginal cost, and every firm's production level becomes infinitesimal in accordance with (3.23).

Let us be more precise. Define the average cost of a firm as

$$AC_i(q_i) = \frac{C_i(q_i)}{q_i},$$

taking into account that $C_i(0) = 0$ and applying l'Hôpital rule,

$$\lim_{q_i \to 0} AC_i(q_i) = C_i'(0),$$

so that as $q_i \to 0$ (i.e. $n \to \infty$), average cost approaches marginal cost.

Finally, recall that a perfectly competitive equilibrium is characterized by the equality between price and marginal cost and also between marginal cost and the minimum of the average cost. Then,

Proposition 3.3. *Consider a homogeneous product industry satisfying assumptions 3.9, 3.10, and 3.11. Then the Cournot equilibrium converges towards the long run competitive equilibrium if* $C_i'(0) = \min AC_i(q_i),\ \forall i$.

Proposition 3.4. *If* $C_i(q_i)$*is* U*-shaped, and* $\exists i$ *s.t.* $C_i'(0) > AC_i(q_i)$*, then the Cournot equilibrium does not converge towards the competitive equilibrium.*

To illustrate these results, lets take our earlier example again:

$$C_i(q_i) = cq_i,\ c > 0,\ i = 1, 2, \dots, n.$$
$$f(Q) = \begin{cases} a - b\sum_i q_i & \text{if} \sum_i q_i \leq \frac{a}{b} = \overline{Q}, \quad a, b > 0, \\ 0 & \text{if} \sum_i q_i > \overline{Q} \end{cases}$$

The Cournot equilibrium is,

$$\begin{aligned} q_i^* &= \frac{a-c}{(n+1)b} \\ Q^* &= \frac{n(a-c)}{(n+1)b} \\ P^* &= \frac{a+nc}{n+1} \\ \Pi_i^*(q^*) &= \frac{(a-c)^2}{(n+1)^2 b},\ i = 1, 2, \dots, n. \end{aligned}$$

Differentiating these equilibrium values with respect to n we obtain,

$$\frac{\partial q_i^*}{\partial n} < 0, \quad \frac{\partial Q^*}{\partial n} > 0,$$
$$\frac{\partial P^*}{\partial n} < 0, \quad \frac{\partial \Pi_i^*}{\partial n} < 0.$$

Moreover,

$$\lim_{n\to\infty} q_i^* = 0,$$
$$\lim_{n\to\infty} Q^* = \frac{a-c}{b}\left(< \frac{a}{b} = \overline{Q}\right),$$
$$\lim_{n\to\infty} P^* = c,$$
$$\lim_{n\to\infty} \Pi_i^* = 0.$$

Thus, in this example the Cournot equilibrium is quasi-competitive. Note also,

$$C_i^{'}(q_i) = c = AC_i(q_i),$$

so that ,

$$c = \lim_{n\to\infty} P^* = \lim_{q_i\to 0} C_i^{'}(q_i^*) = \lim_{q_i\to 0} AC_i^{'}(q_i^*) = \min_{q_i} AC_i(q_i) \ \ \forall i.$$

That is the Cournot equilibrium converges towards the competitive equilibrium.
Finally, note that assumptions 3.9, 3.10, and 3.11 hold:

$$G = b < \infty,$$
$$\overline{Q} = \frac{a}{b} < \infty,$$
$$C_i(0) = 0,$$
$$C_i^{'}(q_i) > 0.$$

Comparing Cournot, monopoly and competitive solutions.

To complete the discussion we can compare the equilibrium output levels under symmetric Cournot oligopoly, monopoly, and perfect competition. In the space of production plans, we can represent the reaction functions and the combinations of output volumes that together give rise to the monopoly (Q^M) and competitive (Q^C) output levels. Figure 3.12 shows them.

It turns out that the aggregate Cournot output ($q_1^N + q_2^N = Q^N$) is an intermediate value between the competitive and monopoly equilibrium outputs. Formally,

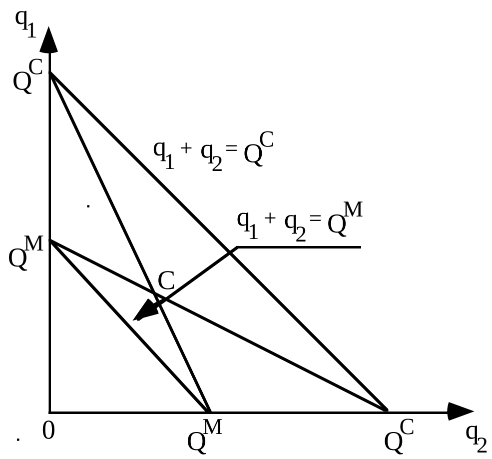

Figure 3.12: Cournot, monopoly and perfect competition equilibria.

Proposition 3.5. *Consider a symmetric duopoly where $C_1' = C_2' = c$. Then, the equilibrium Cournot price, p^N is greater than the competitive price, c, and smaller than the monopoly price p^m.*

Proof. (i) $p^N > c$.

Given that for every firm the first order condition is verified,

$$f'(q_1^N + q_2^N)q_1^N + f(q_1^N + q_2^N) = c,$$
$$f'(q_1^N + q_2^N)q_2^N + f(q_1^N + q_2^N) = c.$$

Adding up these equalities we obtain,

$$\frac{1}{2}f'(q_1^N + q_2^N)(q_1^N + q_2^N) + f(q_1^N + q_2^N) = c, \tag{3.26}$$

where $(q_1^N + q_2^N) > 0$. Since $f'(\cdot) < 0$, it follows that $f(q_1^N + q_2^N) > c$, as required.

(ii) $p^N < p^m$.

We want to show that $(q_1^N + q_2^N) > q^m$ (i.e. $f(q_1^N + q_2^N) < f(q^m)$). This argument will be developed in two steps.

(iia) We first show, by contradiction, that $(q_1^N + q_2^N) \geq q^m$.
Assume $(q_1^N + q_2^N) < q^m$. Then firm j, $j = 1, 2$ can increase its production level to $\widehat{q}_j = q^m - q_k^N$, $(k = 1, 2,\ k \neq j)$, so that joint profits would increase (and would equal monopoly profits).

Also, the increase in production by firm j increases the aggregate production level. Thus, price must decrease. Accordingly, firm k must be worse off (since it is selling the same output, q_k^N at a lower price) while firm j is better off. In other words, firm j would have a profitable deviation which would be a contradiction with the fact that (q_1^N, q_2^N) are equilibrium output levels. Therefore it has to be the case that $(q_1^N + q_2^N) \geq q^m$

(iib) We will show now $(q_1^N + q_2^N) \neq q^m$.

Since (q_1^N, q_2^N) is an equilibrium production plan, condition (3.26) must hold. Assume $(q_1^N + q_2^N) = q^m$. From (3.26) we obtain,

$$\frac{1}{2} f'(q^m) q^m + f(q^m) = c,$$

but this violates the first order condition of the monopolist's profit maximization problem. Therefore, it must be the case that $(q_1^N + q_2^N) \neq q^m$.

Putting together (iia) and (iib) yields $(q_1^N + q_2^N) > q^m$, and proves the result.

□

3.1.11 Stability of the Cournot equilibrium.

The model Cournot proposed is a static model. All actions taken by the agents in the market are taken simultaneously. One way to understand this timing of actions is either that the model is timeless (agents do not have memory) or that all actions are taken in the same time period (there is neither past nor future). In any case, the model does not have a history of actions. Even though, in this type of models it is common to study the stability of the equilibria that may appear. Cournot himself initiated this analysis that Walras followed in his study of the stability of the (static) general equilibrium model.

Although this analysis is fairly popular, it contains a contradiction because the stability is by definition a *dynamic* property. The way to overcome this difficulty is to introduce some dynamic assumptions "ad hoc" in the static model. This technique has been the source of great confusion in the analysis of oligopoly models.

Stability in a duopoly Cournot model.

Consider a homogeneous product duopolistic industry with the following market demand and technologies:

$$C_1(q_1) = 6000 + 16q_1,$$
$$C_2(q_2) = 9000 + 10q_2,$$
$$P = 100 - 0.1(q_1 + q_2).$$

Now we introduce fictitious time. We assume that in every period t, $t = 1, 2, 3, \ldots$ each firm recalls the decisions taken by itself and its rival in the previous period $t - 1$[6].

In period t, firm j expects that its rival, firm i will maintain the same output as in the previous period, $q^e_{it} = q_{it-1}, i = 1, 2$. Every firm in each period aims at maximizing profits of that period. This means that each firm i decides the optimal output level in period t as a function of the observed behavior of the rival, i.e. its decision in $t - 1$. Profit functions are given by,

$$\Pi_1(q_{1t}, q_{2t-1}) = 84q_{1t} - 0.1q^2_{1t} - 0.1q_{1t}q_{2t-1} - 6000,$$
$$\Pi_2(q_{1t-1}, q_{2t}) = 90q_{2t} - 0.1q^2_{2t} - 0.1q_{2t}q_{1t-1} - 9000.$$

The system of first order conditions yield the reaction functions:

$$q_{1t} = 420 - 0.5q_{2t-1},$$
$$q_{2t} = 450 - 0.5q_{1t-1}.$$

These reaction functions allow us to study how output levels evolve with t. Assume that in $t = 0$ the production plan correspond to the static Cournot equilibrium $(q^c_1, q^c_2) = (260, 320)$. Then, in $t = 1$ there is a shock so that firms decisions are $(q_{11}, q_{21}) = (100, 800)$. Using the reaction functions we can compute the temporal evolution of the production levels:

[6]The classic reference in stability analysis is Hahn (1962). Seade (1977) generalizes Hahn's results.

$$
\begin{aligned}
t = 0: &\ (q_1^c, q_2^c) = (260, 320),\\
t = 1: &\ (q_{11}, q_{21}) = (100, 800),\\
t = 2: &\ (q_{12}, q_{22}) = (20, 400),\\
t = 3: &\ (q_{13}, q_{23}) = (220, 440),\\
t = 4: &\ (q_{14}, q_{24}) = (200, 340),\\
t = 5: &\ (q_{15}, q_{25}) = (250, 350),\\
t = 6: &\ (q_{16}, q_{26}) = (245, 325),\\
t = 7: &\ (q_{17}, q_{27}) = (257.5, 327.5),\\
t = 8: &\ (q_{18}, q_{28}) = (256.25, 321.25),\\
t = 9: &\ (q_{19}, q_{29}) = (259.375, 321.875),\\
t = 10: &\ (q_{110}, q_{210}) = (259.0625, 320.3125),\\
&\ \vdots\\
&\ (q_{1t}, q_{2t}) \overset{t\to\infty}{\longrightarrow} (260, 320).
\end{aligned}
$$

Therefore, we see that as time goes by production plans converge towards the static Cournot equilibrium.

In general, with n firms in the market we have a system of first order conditions,

$$q_{it} = w_i(q_{t-1}^{-i}),\ i = 1, 2, \dots, n,$$

where q_{t-1}^{-i} denotes a $n-1$ dimensional production plans of all firms except firm i in $t-1$.

Definition 3.11 (Stable equilibrium)**.** *Let $q^c \in \mathbf{R^n}$ be a static Cournot equilibrium production plan. Let $q_0 = (q_{10}, q_{20}, \dots, q_{n0})$ be an arbitrary production plan. We say that q^c is a stable equilibrium production plan if the sequence of production plans $\{q_t\}_{t=1}^{\infty},\ q_t = (q_{1t}, q_{2t}, \dots, q_{nt})$ converges towards q^c. In other words, if $\lim_{t\to\infty} q_t = q^c$.*

A *sufficient condition* to guarantee the stability of a Cournot equilibrium is that all reaction functions $w_i,\ i = 1, 2, \dots, n$ be *contractions*.

Definition 3.12 (Contraction)**.** *Let f be a continuous function defined on $[a, b]$. Consider two arbitrary points $x, y \in [a, b]$. We say that f is a contraction if*

$$\Big|f(x) - f(y)\Big| \leq c\Big|x - y\Big| \quad \forall x, y \in [a, b],\ c < 1$$

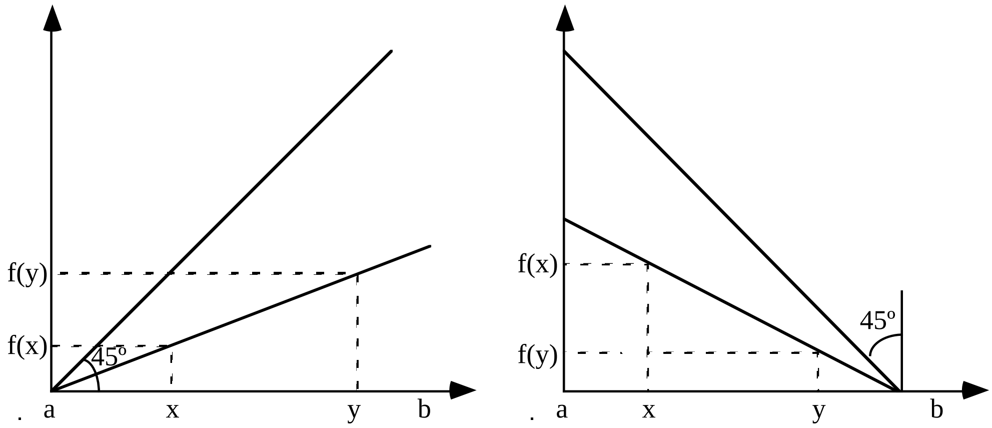

Figure 3.13: Examples where f is a contraction.

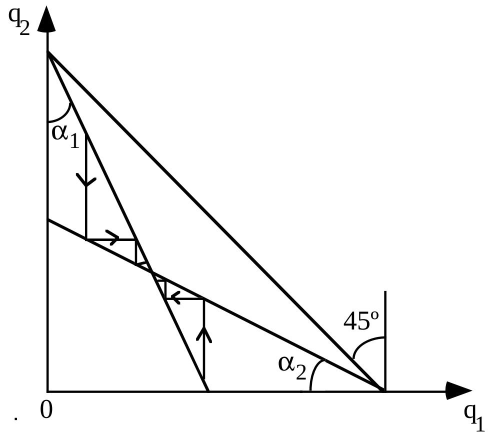

Figure 3.14: Stable equilibrium.

In words, f is a contraction if given two arbitrary points in the domain of the function, the distance between their images is smaller than the distance between the points. If f is linear this simply means that the slope has to be smaller than one. Figure 3.13 illustrates this concept.

Figure 3.14 shows an example where both reaction functions are contractions. Hence, the equilibrium is stable. Figure 3.15 illustrates a situation where only one of the reaction functions is a contraction and the equilibrium is not stable.

This stability analysis presents two serious objections linked with the construction of the system of reaction functions.

(a) From an economic perspective, it does not make any sense to assume that firms are so myopic to ignore the flow of future profits when deciding today's production level.

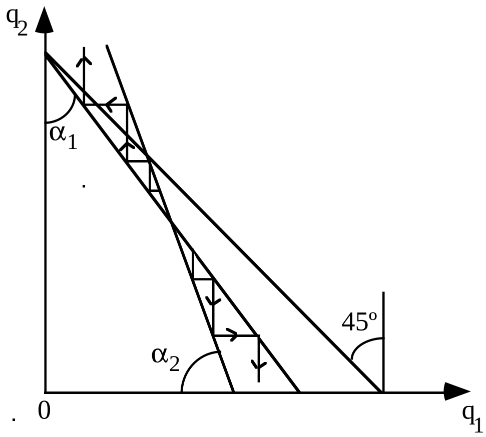

Figure 3.15: Unstable equilibrium.

(b) In the same spirit, it does not make sense either to assume that a firm expect that its rivals will not vary their decisions from yesterday, in particular when our firm is changing its decision in every period (see example).

Note that this objections refer to the formation of expectations, i.e. to the construction of the reaction functions, but not to the concept of Cournot equilibrium.

A more general analysis of the stability of the Cournot equilibrium can be found in Okuguchi (1976), pp. 9-17.

3.2 Price competition.

3.2.1 Introduction.

Cournot's model caught the general attention of the profession 45 years after, in 1883, when a french mathematician Joseph Bertrand published a critical appraisal of Cournot's book.

Bertrand's main criticism is to consider that the *obvious* outcome of Cournot's analysis is that oligopolists will end up colluding in prices, a behavior ruled out by Cournot. Bertrand sets up a variation a Cournot's model where firms take prices as strategic variable. To justify this change of strategic variable, Bertrand argues that in a scenario with perfect and complete information, homogeneous product, without transport costs, and constant marginal costs, every consumer will decide to buy at the outlet with the lowest price.

Actually, Bertrand's point goes beyond. If we assume that firms choose quantities, it not specified in Cournot's model what mechanism determines prices. In a perfectly competitive market, it is irrelevant what variables is decided upon because Smith's "invisible hand" makes the markets clear. In oligopoly, there is no such device. Therefore, a different mechanism is needed to determine the price that, given the production of the firms allow the markets to clear. Accordingly, it may be more reasonable to assume that firm decide prices and production is either sold in the market or stocked.

Thus, Bertrand's model solves one institutional difficulty, but rises another difficulty. In the real world it is difficult to find homogeneous product markets. More often than not, we observe apparently stable markets where different firms sell their products at different prices and all of them obtain positive market shares. In these markets slight variations of prices generate just slight modifications of market shares rather than the bankruptcy of the firm quoting the highest price.

Oligopoly models of homogeneous product seem to contain a dilemma. Either we consider Cournot's model that behaves in a reasonable way but uses the wrong strategic variable, or we consider Bertrand's model where the "good" strategic variable is chosen but, as we will see below, behaves in a degenerated way. This is the so-called Bertrand paradox. After studying Bertrand's model we will also examine some proposals to scape from this paradox.

3.2.2 Bertrand's model.

Let us consider a n firm industry where firms produce a homogeneous product using the same constant marginal cost technology, $C_i(q_i) = cq_i \ \ \forall i$. Consumers behavior is described by a (direct) demand function, $Q = f(P)$ satisfying all the necessary properties.

To determine markets shares, Bertrand assumes the following:

Assumption 3.12 (Sharing rule)**.**

- *the firm deciding the lowest price, gets all the demand ($P_i < P_{-i} \Longrightarrow D_j(P_i, P_{-i}) = 0, \ j \neq i$)[7];*
- *if all firms decide the same price, they share demand evenly ($P_i = P_j, \ \forall j \neq i \Longrightarrow D_i(P_i, P_{-i}) = D_j(P_i, P_{-i}), \ j \neq i$); This is a particular sharing rule based on the symmetry of the model. A possible alternative sharing rule could be to decide randomly which firm gets all the market (see Hoerning (2005) and Vives (1998, ch. 5)).*
- *consumers have reservation prices sufficiently high so that they are all served regardless of the prices decided by firms. To ease computations, we normalize the size of the market to the unit, that is $\sum_{i=1}^{n} D_i(P_i, P_{-i}) = 1$.*

[7] We are abusing notation here. P_i denotes firm i's price, while P_{-i} is a $n-1$ dimensional vector of prices of all firms but firm i.

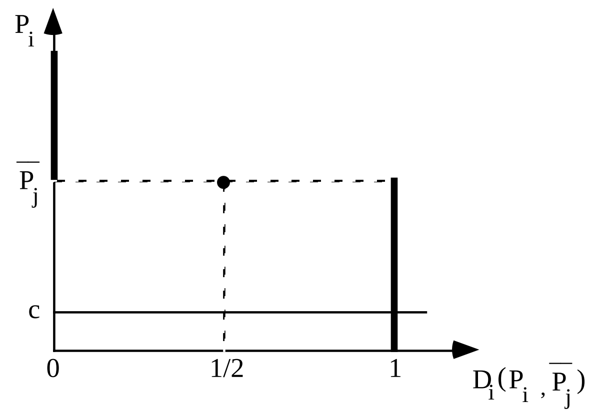

Figure 3.16: Firm i's contingent demand.

This assumption allows to define firm i's contingent demand as

$$D_i(P_i, P_{-i}) = \begin{cases} 0 & \text{if } P_i > P_j,\ \forall j \neq i, \\ \frac{1}{n} & \text{if } P_i = P_j,\ \forall j \neq i, \\ 1 & \text{if } P_i < P_j,\ \forall j \neq i. \end{cases}$$

This represents firm i's market share contingent on its conjecture about the behavior of its competitors (P_{-i}). Figure 3.16 illustrates firm i's contingent demand for a duopolistic market, where P_{-i} reduces to $\overline{P}_j$ the expectation on the behavior of the rival firm.

The system of contingent demand functions allows to define the corresponding system of contingent profit functions, as

$$\Pi_i(P_i, P_{-i}) = \begin{cases} 0 & \text{if } P_i > P_j,\ \forall j \neq i, \\ (P_i - c)\frac{1}{n} & \text{if } P_i = P_j,\ \forall j \neq i, \\ (P_i - c) & \text{si } P_i < P_j,\ \forall j \neq i. \end{cases}$$

Figure 3.17 illustrates firm i's contingent profits for a duopolistic market.,

Now, we can define the equilibrium concept.

Definition 3.13. *A n-dimensional vector of prices (P_i, P_{-i}) is a Bertrand (Nash) equilibrium if and only if*

$$\forall i, \forall P_i \quad \Pi_i(P_i^*, P_{-i}^*) \geq \Pi_i(P_i, P_{-i}^*)$$

Proposition 3.6. *Let us consider a n firm industry where firms produce a homogeneous product using the same constant marginal cost technology, $C_i(q_i) = cq_i \ \forall i$. Let us normalize the size of the market to the unit and assume consumers have sufficiently high reservation prices. Then, there is a unique Bertrand equilibrium given by $P_i^* = c \, \forall i$.*

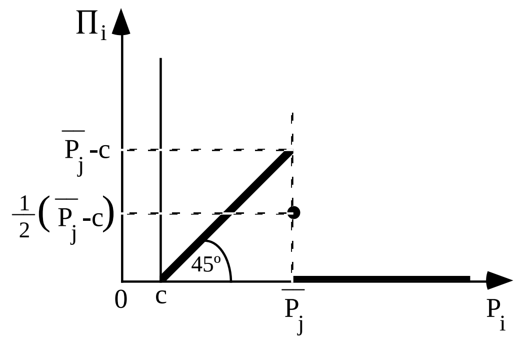

Figure 3.17: Firm i's contingent profits.

Proof. We will develop the argument of the proof for the case of duopoly. It trivially generalizes to n competitors.

Consider an alternative price vector $(\widetilde{P}_i, \widetilde{P}_j)$.

- if $\widetilde{P}_i < \widetilde{P}_j \Longrightarrow D_j(\widetilde{P}_i, \widetilde{P}_j) = 0$ and also, $\Pi_j(\widetilde{P}_i, \widetilde{P}_j) = 0$. Firm j can nevertheless improve upon its profits by choosing a price $P_j < \widetilde{P}_i$. Therefore, $(\widetilde{P}_i, \widetilde{P}_j)$ such that $\widetilde{P}_i < \widetilde{P}_j$ cannot be an equilibrium price vector;

- if $\widetilde{P}_i > \widetilde{P}_j \Longrightarrow D_i(\widetilde{P}_i, \widetilde{P}_j) = 0$ and also $\Pi_i(\widetilde{P}_i, \widetilde{P}_j) = 0$. Firm i can nevertheless improve upon its profits by choosing a price $P_i < \widetilde{P}_j$. Therefore, $(\widetilde{P}_i, \widetilde{P}_j)$ such that $\widetilde{P}_i > \widetilde{P}_j$ cannot be an equilibrium price vector;

- from the previous arguments it follows that if there is an equilibrium, it has to satisfy $\widetilde{P}_i = \widetilde{P}_j$. Thus, let us consider a price vector $(\widetilde{P}_i, \widetilde{P}_j)$ such that $\widetilde{P}_i = \widetilde{P}_j > c$.

  Now, $D_i(\widetilde{P}_i, \widetilde{P}_j) = D_j(\widetilde{P}_i, \widetilde{P}_j) = \frac{1}{2}$ and $\Pi_i(\widetilde{P}_i, \widetilde{P}_j) = \Pi_j(\widetilde{P}_i, \widetilde{P}_j) = \frac{1}{2}(\widetilde{P}_i - c)$. Given that $\widetilde{P}_i = \widetilde{P}_j > c$, firm i has a profitable unilateral deviation $\widetilde{P}_i - \varepsilon$ because its profit increases to $\Pi_i(\widetilde{P}_i - \varepsilon, \widetilde{P}_j) = \widetilde{P}_i - \varepsilon - c > \frac{1}{2}(\widetilde{P}_i - c)$, for ε sufficiently small. A parallel argument is also true for firm j. Hence, the two firms start a price war decreasing their respective prices so that $(\widetilde{P}_i, \widetilde{P}_j)$ such that $\widetilde{P}_i = \widetilde{P}_j > c$ cannot be an equilibrium either;

- finally assume $\widetilde{P}_i = \widetilde{P}_j = c$. Now, $D_i(\widetilde{P}_i, \widetilde{P}_j) = D_j(\widetilde{P}_i, \widetilde{P}_j) = \frac{1}{2}$ and $\Pi_i(\widetilde{P}_i, \widetilde{P}_j) = \Pi_j(\widetilde{P}_i, \widetilde{P}_j) = 0$. In this situation no firm has a profitable unilateral deviation. An increase in the price yields zero profit; a decrease in the price yields negative profits.

Therefore, we conclude that a price vector $(\widetilde{P}_i, \widetilde{P}_j)$ such that $\widetilde{P}_i = \widetilde{P}_j = c$ is the only Bertrand equilibrium of this game. $\square$

We can also illustrate this argument in terms of the reaction functions. Let P^m denote the monopoly price and study firm i's the best reply to any conjecture of the price of firm j.

- If $P_j > P^m$, firm i's best reply is to choose the monopoly price to obtain monopoly profits.
- If $P_j < c$, firm i's best reply is to choose a price equal to the marginal cost to obtain zero profits. Actually, any price $P_i > P_j$ yields zero profit to firm i, so that the reaction function becomes a correspondence.
- If $c < P_j < P^m$ we have to distinguish three cases.
  (a) If $P_i > P_j$, then $\Pi_i = 0$;
  (b) If $P_i = P_j$, then $\Pi_i = (P_i - c)\frac{1}{2}$;
  (c) If $P_i < P_j$, then $\Pi_i = (P_i - c)$. In this case the profit function is increasing in P_i, so that firm i's best reply is the highest possible price, that is $P_i = P_j - \varepsilon$, for ε arbitrarily small.

Summarizing, firm i's reaction function is,

$$P_i^*(P_j) = \begin{cases} P^m & \text{if } P_j > P^m \\ P_j - \varepsilon & \text{if } c < P_j \leq P^m \\ c & \text{if } P_j \leq c \end{cases}$$

By symmetry, firm j has a similar reaction function exchanging the subindices adequately. Figure 3.18 illustrates them. It is easy to verify that that these reaction functions intersect only at $P_i = P_j = c$, thus characterizing the Bertrand equilibrium of the model.

It is important to note that in this model two firms are enough to obtain the competitive equilibrium. Recall that when firms compete in quantities, the convergence of the Cournot equilibrium towards the competitive equilibrium, when it occurs, requires an arbitrarily large number of firms. Therefore, the nature of the equilibrium according to the strategic variable chosen by firms is very different. In the following sections we will review some of the efforts developed to avoid the price war inherent to Bertrand's model.

3.3 Cournot vs. Bertrand.

We have studied two homogeneous product models of oligopoly and their conclusions are very different. We want to get some better understanding of this different

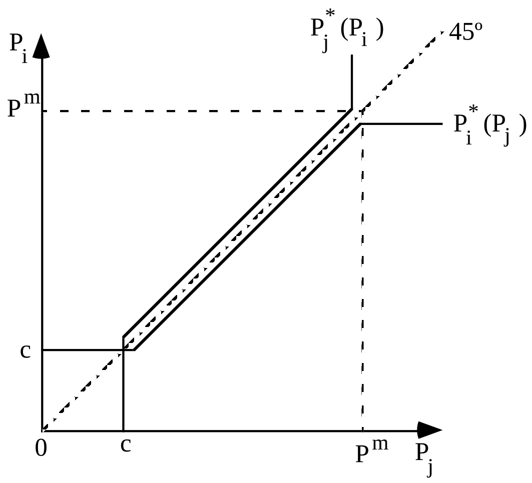

Figure 3.18: Bertrand equilibrium.

behavior. We will see that the basic reason is the different "residual demand" a firm faces under price or quantity competition. We will develop the argument in terms of a duopoly for simplicity. It generalizes trivially to n firms.

Thus, consider a homogeneous product duopoly with demand BBB as in figure 3.19.

Assume firms compete in quantities and that firm 1 conjectures that firm 2 will choose a production volume q_2. Conditional on this expectation, firm 1 faces a residual demand given by ACC.

Assume now firms competing in prices, and firm 1 conjectures that firm 2 will choose a price $P^* = P(q_1 + q_2)$ (where, for comparison purposes, $(q_1 + q_2)$ is the same aggregate output as in the previous quantity competition). Now firm 1's contingent residual demand is given by BP^*BB.

Accordingly, in general we should expect firm 1 facing a more elastic contingent residual demand under price competition than under quantity competition. Therefore, we should also expect lower equilibrium prices, higher aggregate production levels and thus better performance under Bertrand than under Cournot behavior.

When firms compete in production levels, every firm knows that its competitor has committed to a certain output. When firms compete in prices, they know that undercutting on the rival's price yields the whole market. Thus, firms are more aggressive, driving prices down.

We have argued before about the "properness" of the price as strategic variable, but the paradoxical behavior that it conveys in markets of homogeneous product. Several attempts have been proposed to obtain a "normal" behavior of the market maintaining the assumption of homogeneous product. These can be

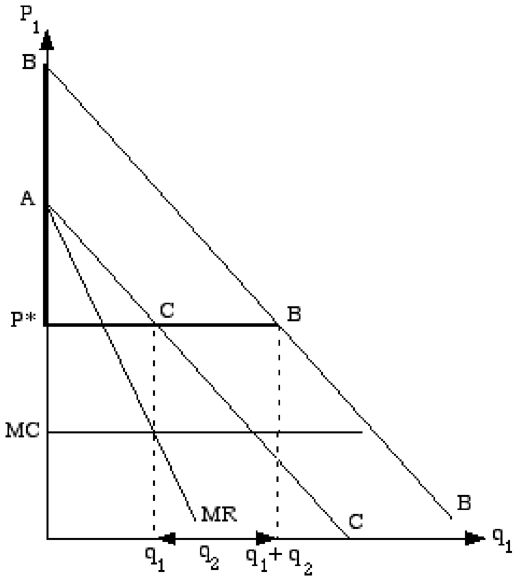

Figure 3.19: Cournot vs. Bertrand.

grouped generically in six categories:

Capacity constraints We find models where restrictions are imposed on the technology of the firms to control for the price wars generated by the constant marginal costs technology. This family of models assume decreasing returns to scale technologies, that is some form of capacity constraints. This implies that a firm may not find profitable to serve all its demand. Two phenomena arise. On the one hand price wars do not appear. On the other hand rationing comes into play. We will focus on two models as examples of (i) exogenous capacity constraints (Edgeworth) and (ii) endogenous capacity constraints (Kreps and Scheinkman).

Contestability The departure point is the attempt to generalize the theory of perfectly competitive markets by endogenizing the structure of the market. If a perfectly competitive outcome can be supported as an equilibrium, then oligopolistic behavior could be though of as determined by the pressure of potential competition. The distinctive feature of a contestable market is that the capacity of a potential entrant to rip off all positive profits. This maintains the incumbent firm at the competitive equilibrium.

Price rigidities A different approach focusses on the perfect flexibility of prices. Sweezy observes that in real markets prices are flexible when increasing but much less flexible when decreasing. In accordance with this observation he proposes a model with price rigidities.

Commitment The idea of commitment is introduced in the form of a two-stage decision process where one firm takes its decision prior to the other firms in the market. This commitment then is illustrated as the market leadership position of one firm over its competitors.

Conjectual variations A Nash equilibrium (be it in prices or quantities) is characterized by the set of reaction functions. In general, the interdependence among firms' decisions is captured by the effect of one firm's decision on the aggregate output and thus, on its rivals' behavior. This effects is the so-called conjectural variation. Different assumptions on the way each firm makes conjectures of its rivals' behavior lead to different equilibrium configurations.

Dynamic models

3.4 Variations 1. Capacity constraints.

In this section we analyze models that focus of the technology as a way to avoid the price wars arising under price competition when firms exhibit constant returns to scale (constant marginal costs). Generically, these models will assume decreasing returns to scale in the form of strictly convex costs leading to capacity constraints.

Edgeworth (1897) proposed an extreme form of exogenous capacity constraint by assuming that firms had technologies described by a constant marginal cost up to a certain production level and infinite beyond. We will see though that this model does not yield any equilibrium. Again for simplicity, we will illustrate the model for the case of duopoly. As in the previous occasions, the argument generalizes trivially to an arbitrary number of firms.

The second model is due to Kreps and Scheinkman (1983). They propose a model with endogenous capacity constraints. In particular, they propose a two-stage game where firms (simultaneously) commit to output levels and compete in prices in the second stage. The surprising result is that the subgame perfect equilibrium of the game yields the Cournot production levels.

3.4.1 Rationing rules.

Before going into the analysis of these models, we have already mentioned that one consequence of introducing constraints in the capacity of production of firms is that, at a given price a firm faces excess demand so that not all consumers can be attended. In other words, there will be rationing in the market. Thus, an issue to be tackled is what consumers are going to be served by the firm.

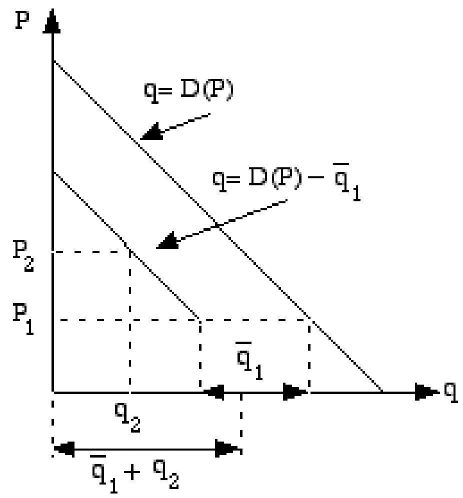

Figure 3.20: The efficient rationing rule.

There are two popular rationing rules: The efficient and the proportional rationing rules. As usual, to illustrate let us consider a duopoly where $P_1 < P_2$ and $\overline{q}_1 \equiv S(P_1) < D(P_1)$.

Efficient rationing rule This rule corresponds to a "first come, first served" rule. That is, the firm attends the most eager consumers and firm 2 serves the rest. Formally,

$$D_1(P_1) = \overline{q}_1$$
$$D_2(P_2) = \begin{cases} D(P_2) - \overline{q}_1 \text{ if } D(P_2) > \overline{q}_1 \\ 0 \text{ otherwise} \end{cases}$$

That is, firm 2's residual demand is the result of shifting the market demand inwards by the amount $\overline{q}_1$. Figure 3.20 illustrates the argument.

This rule is called efficient because it maximizes consumer surplus.

Proportional rationing rule Under this rule, any consumer has the same probability of being rationed. It is a randomized rationing rule. The probability of *not* being able to buy from firm 1 is

$$\frac{D(P_1) - \overline{q}_1}{D(P_1)}.$$

Therefore, the residual demand for firm 2 rotates inwards around the intersection point of the market demand with the price axis. That is, the slope of

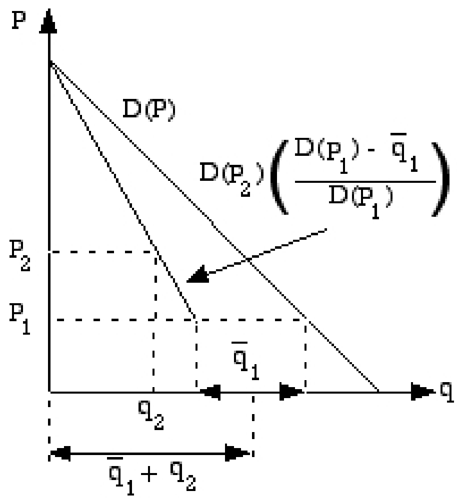

Figure 3.21: The proportional rationing rule.

the residual demand is modified by the probability of buying at firm 2.

$$D_2(P_2) = D(P_2)\Big(\frac{D(P_1) - \overline{q}_1}{D(P_1)}\Big).$$

This rule is not efficient. Consumers with valuations below P_2 may buy the commodity because they find it at a bargain price P_1. Figure 3.21 illustrates the argument.

Finally, we can compare these two rationing rules in terms of firm 2's market share. As it is easy to verify, and figure 3.22 illustrates, at any price the proportional rationing rule yields higher residual demand to firm 2. Accordingly more consumers are served under the proportional rationing rule although consumer surplus is not maximized.

3.4.2 Exogenous capacity constraints: Edgeworth's cycle.

Consider a market with demand $P = 1 - q$. Firms produce a homogeneous product with a constant (zero) marginal cost technology up to a production level $K_i, \ \forall i$. That is, firm i's feasible production set is given by $\{q_i | q_i \leq K_i\}$. To ease the exposition without much loss of generality, let us assume $K_i = K, \ \forall i$ and that the maximum aggregate production $nK < 1$, i.e. maximum aggregate supply can never satisfy all market demand. In other words, the capacity constraint is effective. Given the symmetry of the model, this means that a single firm's market share is below $1/n$, i.e. $K < \frac{1}{n}$.

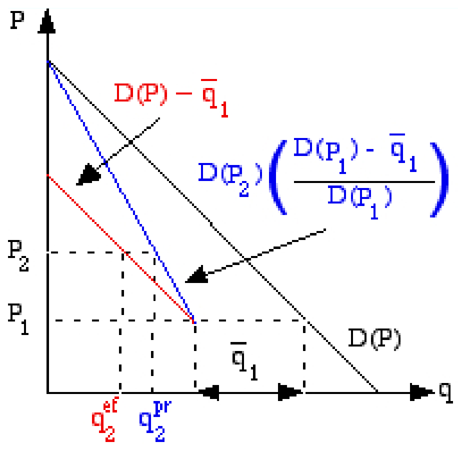

Figure 3.22: Efficient vs. proportional rationing rule.

If all firms collude, together will produce the monopoly output $q^m = \frac{1}{2}$ that will sell at the monopoly price $P^m = \frac{1}{2}$ to obtain an aggregate monopoly profit $\Pi^m = \frac{1}{4}$. The symmetry of the model supports an even distribution of those production and profit levels, i.e., $q_i = \frac{1}{2n}$ and $\Pi_i = \frac{1}{2}\frac{1}{2n} = \frac{1}{4n}$. Thus, assume, as a starting point, $K > \frac{1}{2n}$. The relevant question at this moment is whether this collusive agreement is stable. The answer is negative. Note that a firm i can undercut the monopoly price, $P_i = \frac{1}{2} - \varepsilon$, will satisfy a demand K and will obtain profits $\Pi_i^K = (\frac{1}{2} - \varepsilon)K > \frac{1}{2}\frac{1}{2n}$ for ε sufficiently small. Figure 3.23 illustrates the situation for the duopoly case.

In turn, another firm j may undercut firm i's price to obtain profits $\Pi_j^K = (P_i - \varepsilon)K$. And so on and so forth. How far will this undercutting arrive? To answer we have to study the model with some more detail. To ease the argument, let us assume heretofore that there are only two firms in the industry.

Feasible values for K. Note that when firm i undercuts the price of its rival, and thus produces K, leaves a residual demand $DR = 1 - K$. On this residual demand, the rival firm j obtains monopoly profits (see figure 3.24)

$$\Pi_j^{mK} = \left(\frac{1-K}{2}\right)^2 \tag{3.27}$$

This level of profits imposes a restriction on the value of K. In particular the output level $\frac{1-K}{2}$ must be feasible, that is $\frac{1-K}{2} < K$ or $K > \frac{1}{3}$. Hence, we have a range of feasible values for the capacity constraint to be meaningful: $K \in (\frac{1}{3}, \frac{1}{2})$.

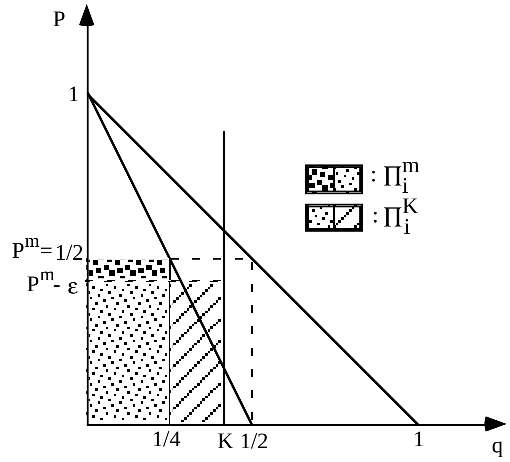

Figure 3.23: Price wars with capacity constraints.

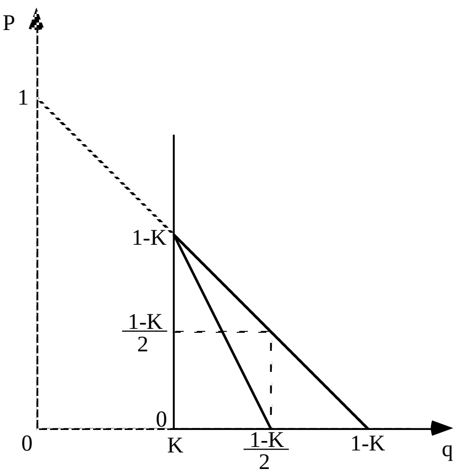

Figure 3.24: Profits over residual demand.

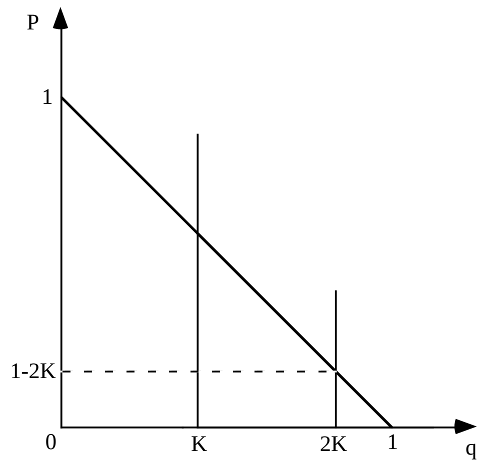

Figure 3.25: Minimum market price.

Minimum market price. If both firms produce their maximum capacities $q_i = K$, the aggregate production $2K$ is sold in the market at the price $P = 1 - 2K$. Given that aggregate production cannot go beyond $2K$, market price cannot fall below $1 - 2K$. Figure 3.25 illustrates.

All the information collected so far is summarized in figure 3.26.

A critical price. Finally, define a price $\widehat{P}$ such that a firm selling K units at this price obtains the same profits as the monopoly profits on the residual demand (3.27), i.e.

$$\widehat{P}K = \left(\frac{1-K}{2}\right)^2 \tag{3.28}$$

This price must be over the minimum market price and under the monopoly price on the residual demand,

$$\widehat{P} \in \left(1 - 2K, \frac{1-K}{2}\right)$$

Now, we can go back to the price war between the firms.

Although Edgeworth model is static, to illustrate the decision process of the firms we will develop the argument in steps. In every step only one firm may undercut its rival, who, in turn, will apply its price on the residual demand.

$\tau = 0$ Take as the initial situation the perfect collusion where firms evenly share monopoly production and profits ($\Pi_i^m = \frac{1}{8}$).

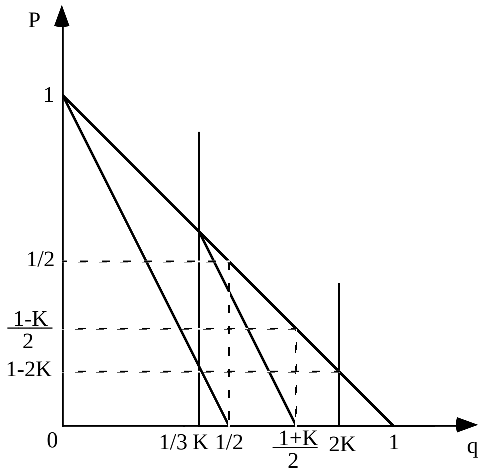

Figure 3.26: Prices and residual demand.

$\tau = 1$ We have already argued that there are incentives for unilateral deviations. Thus, firm i undercuts firm j's price to obtain,

$$\Pi_i^1 = (P^m - \varepsilon)K > \frac{1}{8} \text{ for } \varepsilon \text{ small enough.}$$

In this stage firm j maintains its price P^m on the residual demand to obtain,

$$\Pi_j^1 = P^m RD$$

$\tau = 2$ Now is firm j's turn to act. Firm j has two alternatives:

(a) undercut firm i's price. In this case quotes a price $P_j^2 = P_i^1 - \varepsilon$ to obtain $\Pi_j^2 = (P_i^1 - \varepsilon)K$.

(b) obtain monopoly profits on the residual demand, given by (3.27).

Assume that undercutting is still more profitable.

$\tau = 3$ Now is again firm i's turn to decide. As in the previous stage, firm i has two options:

(a) undercut firm j's price. In this case, it chooses a price $P_i^3 = P_j^2 - \varepsilon$ and obtains $\Pi_i^3 = (P_j^2 - \varepsilon)K$.

(b) obtain monopoly profits on the residual demand, given by (3.27).

This process of undercutting goes on alternatively for each firm, so that the price diminishes at every stage until it arrives at a stage $t = n$ where, say, firm i when undercutting reaches the critical price $\widehat{P}$, i.e.

$\tau = n$ Firm i undercuts and quotes a price $P_i^n = P_j^{n-1} - \varepsilon = \widehat{P}$, so that the distribution of profits in this stage is

$$\Pi_i^n = \widehat{P}K$$
$$\Pi_j^n = P_j^{n-1}RD$$

$\tau = n+1$ Now, once more, firm j faces two alternatives,

(a) undercut firm i's price. In this case, it selects a price $P_j^{n+1} = \widehat{P} - \varepsilon$ and obtains $\Pi_j^{n+1} = (\widehat{P} - \varepsilon)K$.

(b) obtain monopoly profits on the residual demand, given by (3.27).

At this stage though, by definition of $\widehat{P}$, it turns out that

$$\Pi_j^{n+1} = (\widehat{P} - \varepsilon)K < \left(\frac{1-K}{2}\right)^2.$$

Therefore firm j is no longer interested in undercutting firm i's price but prefers to obtain the monopoly profits on the residual demand choosing a price $P_j = \frac{1-K}{2} > \widehat{P}$.

$\tau = n+2$ Firm i undercuts firm j's price , $P_i^{n+2} = \frac{1-K}{2} - \varepsilon$ and the cycle restarts.

To conclude, Edgeworth construction yields a process of price undercutting from $\frac{1-K}{2}$ till $\widehat{P}$. Then the price jumps up from $\widehat{P}$ to $\frac{1-K}{2}$ and the undercutting resumes until $\widehat{P}$. Then, the price jumps up to $\frac{1-K}{2}$, and so on (see figure 3.27), so that for any pair of prices (P_i, P_j) there is always at least one firm willing to unilaterally deviate[8]. In other words, there is no price vector constituting a non-cooperative equilibrium.

3.4.3 Endogenous capacity constraints. Kreps and Scheinkman model.

Kreps and Scheinkman (1983) present a two-stage game where firms, in the first stage, simultaneously decide upon production capacity and in the second stage compete in prices. With this model they capture the fact that firms take long run decisions (capacity) and short run decisions (prices). They show that the capacity levels chosen in the first stage and the price chosen in the second stage are precisely the production and the price that would result in a traditional Cournot model. As usual we solve the model backwards.

[8]See a more general approach in Chowdhury (2005), where an equilibrium where all firms charging a price equal to marginal cost holds.

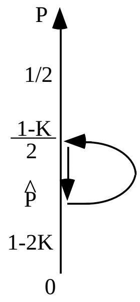

Figure 3.27: Edgeworth cycle.

The model considers two firms with capacity levels $\overline{q}_i,\ i = 1, 2$ that produce a homogeneous product with a technology described by constant (zero) marginal costs up to $\overline{q}_i$. The efficient rationing rule is in action. Finally, market demand is a concave function.

The price game.

We will only characterize the pure strategy equilibrium[9].

Lemma 3.5. *In a pure-strategy equilibrium, $P_1 = P_2 = P(\overline{q}_1 + \overline{q}_2)$. That is, firms sell up to capacity.*

Proof.

- Assume $P_1 = P_2 = P > P(\overline{q}_1 + \overline{q}_2)$. Then the price is too high in the sense that there is at least one firm that cannot sell its capacity, $q_i < \overline{q}_i$. Therefore, choosing $P - \varepsilon$ firm i gets all the market and can sell $\overline{q}_i$. For ε small enough, firm i would find the undercutting profitable, $(p - \varepsilon)\overline{q}_i > pq_i$.

- Assume then, $P_1 = P_2 = P < P(\overline{q}_1 + \overline{q}_2)$. Now both firms ration customers. By slightly raising its price, a firm would still be able to sell its capacity and make more profit, that is, $(p + \varepsilon)\overline{q}_i > p\overline{q}_i$.

  Hence, if in equilibrium $P_1 = P_2$, it has to be the case that, $P_1 = P_2 = P(\overline{q}_1 + \overline{q}_2)$.

- We still have to show that in equilibrium, $P_1 = P_2$. Assume, $P_1 < P_2$. Then firm 1 wants to raise its price if it is capacity constrained. Otherwise, P_1 is firm 1's monopoly price at $c = 0$, and supplies the entire demand; accordingly, firm 2 makes zero profits and has an incentive to undercut to obtain positive profits.

□

[9] Kreps and Scheinkman (1983) also characterize the mixed strategy equilibrium. Tirole (1988, pp. 230-231) provides a sketch of the proof. Boccard and Wauthy (2000) extend the model to the oligopoly case.

Lemma 3.6. *Let $R_i(\overline{q}_j)$ denote firm i's best reply to output $\overline{q}_j$ in the one-shot simultaneous Cournot game without capacity constraints. Then, firm i never charges a price P_i below $P(R_i(\overline{q}_j) + \overline{q}_j)$ inducing to produce beyond the optimal reaction to $\overline{q}_j$*

Proof.

- Let $P_j > P_i$. Then firm i wants to raise its price if it is capacity constrained. Otherwise, P_i is firm 1's monopoly price at $c = 0$, and supplies the entire demand; accordingly, firm j makes zero profits and has an incentive to undercut to obtain positive profits.
- Let $P_j = P_i < P(R_i(\overline{q}_j) + \overline{q}_j)$. First of all, note that in this case, at most one firm has to be capacity constrained. If non would be, then they would have incentives to undercut. Then, if firm i is capacity constrained, it can raise its price and improve on profits $(P_i + \varepsilon)\overline{q}_i > p_i\overline{q}_i$. If firm i is not capacity constrained, necessarily firm j must be. Firm i's residual demand is given by $D(P_i) - \overline{q}_j$ and its profits $\Pi_i = P_i(D(P_i) - \overline{q}_j)$. Since firm i is not capacity constrained we can rewrite its profit as $\Pi_i = P(q_i + \overline{q}_j)q_i$. This is the Cournot profit conditional on $\overline{q}_j$. Therefore, $q_i = R_i(\overline{q}_j)$ by definition of $R_i(\overline{q}_j)$, and $P_i = P(R_i(\overline{q}_j) + \overline{q}_j)$.
- Let $P_j < P_i$. Then, if firm i is capacity constrained, that is, $\overline{q}_i < D(P_i) - \overline{q}_j$, it can raise its price, still sell all its capacity and thus improve upon profits, $(P_i + \varepsilon)\overline{q}_i > p_i\overline{q}_i$. If firm i is not capacity constrained, that is, $\overline{q}_i > D(P_i) - \overline{q}_j$, then its profits are $\Pi_i = P_i(D(P_i) - \overline{q}_j) = P(q_i + \overline{q}_j)q_i$. This is the Cournot profit conditional on $\overline{q}_j$. Therefore, $q_i = R_i(\overline{q}_j)$ by definition of $R_i(\overline{q}_j)$, and $P_i = P(R_i(\overline{q}_j) + \overline{q}_j)$.

□

Finally, we will use lemmas 3.5 and 3.6 to prove the following,

Lemma 3.7. *A pure strategy price equilibrium exists only if $\overline{q}_i \leq R_i(\overline{q}_j),\ \forall i$.*

Proof.

- Assume a price equilibrium exists and $\overline{q}_i > R_i(\overline{q}_j)$.

  From lemma 3.5, we know that $P_i = P(\overline{q}_j + \overline{q}_j)$. Then if $\overline{q}_i \leq R_i(\overline{q}_j),\ \forall i$, it follows that $P_i < P(R_i(\overline{q}_j) + \overline{q}_j)$ contradicting lemma 3.6. Hence, if $\overline{q}_i > R_i(\overline{q}_j)$ there is no pure strategy price equilibrium.
- If $\overline{q}_i \leq R_i(\overline{q}_j)$, then $P_1 = P_2 = P(\overline{q}_1 + \overline{q}_2)$ is an equilibrium. This is so because lowering the price does not allow to sell more; and raising the price means that the quantity sold is below the optimal reaction.

  In particular, if $q_i^*,\ i = 1, 2$ are the Cournot production levels (at a marginal cost c), then the equilibrium price is $P(q_1^* + q_2^*)$.

□

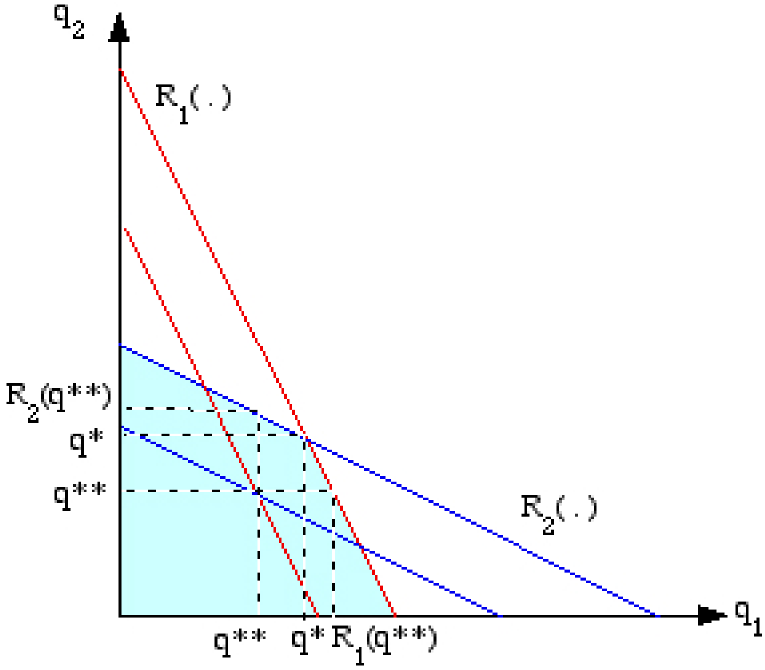

Figure 3.28: Reaction functions

Summarizing, we have shown that a price equilibrium (in pure strategies) exists if and only if capacities are not too high. In equilibrium, $P_1 = P_2 = P(\overline{q}_1 + \overline{q}_2)$. That is, the price at which demand equals aggregate capacity.

The capacity game.

Let $c_0 > 0$ denote the unit cost of installing capacity. We will show that $\overline{q}_1 = \overline{q}_2 = q^{**}$, where $q^{**} = \arg\max q[p(q + q^{**}) - c_0]$ is an equilibrium[10].

In the second stage price game, the capacity cost is sunk and thus irrelevant. Each firm would like to put more output in the market than it would if capacity were yet to be paid for. This implies that from the first to the second stage of the game, reaction curves move upwards, i.e. $R(q^{**}) > q^{**}$, where $R(q^{**})$ denotes the second stage reaction function, and q^{**} is (as defined above) the best first stage reaction to q^{**}. Figure 3.28 illustrates the argument.

Suppose that firm i chooses q^{**}. If firm j plays $q \leq R(q^{**})$ obtains profits

$$q\Big[P(q + q^{**}) - c_0\Big] \leq q^{**}\Big[P(2q^{**}) - c_0\Big].$$

[10] Recall that we are assuming $c = 0$.

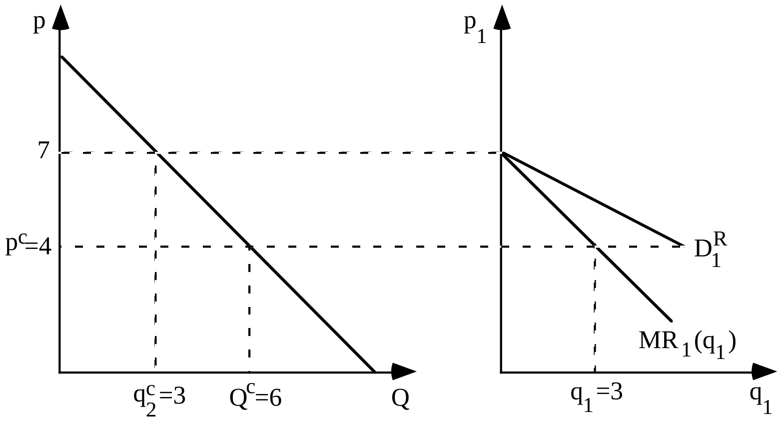

Figure 3.29: Residual demand

Hence, the Cournot equilibrium with cost c_0 is the equilibrium of the first stage of the game.

From the analysis of the price game, the equilibrium price is $P(2q^{**})$.

An example

To illustrate the contents of the model, let us consider the following example. Market demand is given by $P = 10 - Q$, and the unit cost $c_0 = 1$.

Second stage subgame. Assume that for some reason, firms decide the Cournot capacities $q_1^c = q_2^c = 3$, so that $Q^c = 6$.

We want to show that firms choose prices that clear the market, given Cournot capacities, that is $P^c = 4$.

Consider a deviation by firm 1, given $P_2 = 4$.

- If $P_1 < P_2 = 4$, given that firm 1 is already selling its capacity, profits cannot increase.
- If $P_1 > P_2 = 4$, firm 2 sells its capacity $q_2 = 3$ and firm 1 becomes a monopolist on the residual demand. It is given by $D_1^R = 7 - q_1$. Its associated marginal revenue is $MR_1(q_i) = 7 - 2q_1$. Given that $MR_1(q_1) > 0$ for all $q_1 < \frac{7}{2}$, it follows that the residual demand in that range of values is elastic. Accordingly, an increase in the price will lower the revenues of the firm.

Figure 3.29 illustrates the argument.

Therefore, $P_1 = P_2 = 4$ is the only price equilibrium of the second stage game for $Q^c = 6$.

First stage subgame. Firms when deciding capacities, anticipate that in the price game firms will choose the prices that will clear the market. Accordingly, the problem of the choice of capacities is equivalent to the problem of selecting production levels in a traditional Cournot model. Hence, $q_1^c = q_2^c = 3$.

To complete the analysis two comments are in order. First the result of the model is not general. Second, we have only considered output levels such that $q_1 + q_2 \leq 6$. The full proof requires also the examination of the case $q_1 + q_2 > 6$. But this involves the use of mixed strategies.

More general results are provided by Vives (1993) and Boccard and Wauthy (2000). In the same spirit, Maskin (1986) and Friedman (1988) study the xcase of durable goods and Judd (1990) the case of perishable goods. Two more approaches are worth mentioning. Dasgupta and Maskin (1986) and Maskin (1986) show that this type of games yield no equilibrium in pure strategies when firms' decisions on prices and quantities are simultaneous. Chowdhury (2005) compares the simultaneous and sequential decisions. Finally, Grossman (1981) and Hart (1982) propose the so-called “supply function” equilibrium where firms' strategies consist in complete profiles of price-quantity pairs.

3.5 Variations 2. Contestable markets.

The outcome of Bertrand model where two firms are enough to obtain the competitive solution, make wonder about the possibility to generalize the theory of perfectly competitive markets by endogenizing the determination of the structure of the market. Should this attempt be successful, one could conclude that the conjectural variations of potential entrants in a market would not be the crucial element. Rather oligopolistic behavior would be determined by the preassure of potential competition.

The idea of contestable markets appears as the result of the efforts of Bailey, Baumol, Panzar i Willig[11] during the eighties to extend the theory of perfectly competitive markets to situations where scale economies are relevant.

We say that a market is contestable when entry is free and exit is costless, potential entrants have access to the same technology as incumbents, and these potential entrants evaluate the profitability of entry taking as reference the prices of the incumbents *before* entry takes place. In other words, potential entrants think that they can undercut incumbents and “steal” all the demand before the incumbents will react. Thus, a contestable market appears when it is vulnerable to a process of “hit-and-run” entry.

[11]Bailey (1981,1982), Bailey i Baumol (1984) Bailey i Panzar (1981), Baumol (1982), Baumol i Willig (1986), Baumol, Panzar i Willig (1982,1986).

The distinction between a contestable and a perfectly competitive market relies in that the perfectly competitive market assumes firms without capacity to affect the market price, while in a contestable market, both incumbents and potential entrants are aware that, demand determines the amount consumers are willing to buy at the market price. But they are also aware that by quoting a price under the market price more production can be sold. Accordingly, in a contestable market there is no need for a large number of firms, even though the opportunity for a potential hit-and-run entrant ensures zero equilibrium profits. Thus, in a contestable market the equilibrium production is always efficient regardless of the number of firms, since price always equals marginal cost (see Bailey and Friedlander (1982), pp. 1039-1042.).

Following Martin (2002 supplement) and Shy (1995) we list the main definitions and results of the theory for the case of a homogeneous industry with n single-product firms.

Definition 3.14 (Industry configuration)**.** *An industry configuration is a vector* $(n, y^1, y2, \dots, y^n, p)$ *where* n *is the number of firms,* y^i *denotes firm* i*'s output, and* p *is the price that clears the market, that is* $Q(p) = \sum_{i=1}^{n} y^i$.

Definition 3.15 (Feasible configuration)**.** *A configuration is feasible if production is sufficient to meet demand, and no firm is losing money.*

Definition 3.16 (Sustainable configuration)**.** *A configuration is sustainable if it is feasible and no potential entrant can cut price and make profit supplying a quantity less than or equal to the quantity demanded at the lower price.*

Definition 3.17 (Perfectly contestable market)**.** *A market is perfectly contestable if sustainability is a necessary equilibrium condition.*

Definition 3.18 (Long-run competitive equilibrium)**.** *A configuration is a long-run competitive equilibrium if it is feasible and there is no output level at which any firm could earn an economic profit at the prevailing price.*

From these definitions, the following results can be proved.

Lemma 3.8. *A long-run competitive equilibrium is sustainable.*

Proof. From definition 3.18, we know that in a long-run competitive equilibrium the associated configuration is feasible, and no alternative output level allows to earn positive profits to any firm. According to definition 3.16 this configuration is sustainable. □

Lemma 3.9. *A sustainable configuration need not be a long-run competitive equilibrium.*

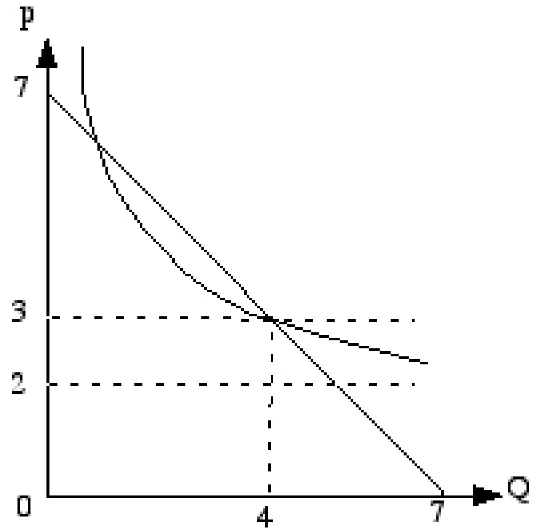

Figure 3.30: Sustainability vs. long-run competitive equilibrium.

Proof. An example will suffice. Let demand be $p = 7 - Q$, and the cost function be $c(q) = 4 + 2q$. Then, any industry configuration $(n, y^1, p) = (1, 4, 3)$ is a sustainable equilibrium. If one firm sells 4 units, average cost is 3, and so is the price clearing the market. Thus, output equals demand, and price equals average cost and the configuration is feasible. Nevertheless, at a price 3 a firm could produce more than 4 units of output and would earn a positive profit since average cost falls below 3 as production increases beyond 4 units. Figure 3.30 illustrates the argument. □

Lemma 3.10. *If a configuration is sustainable, all firms earn zero profit, and the market clearing price is not below marginal cost.*

Proof. If an incumbent firm were getting positive profits, and entrant could mimic the incumbent's output, undercut its price and still obtain positive profits. This would contradict definition 3.16 though. On the other hand, if price were below marginal cost and a firm would obtain zero profit, it could reduce output slightly and get positive profits. This would contradict the first part of the statement. □

Lemma 3.11. *If at least two firms are active in a sustainable configuration, price equals marginal cost for all firms.*

Proof. From lemma 3.10 we already know that price is higher than or equal to marginal cost. If price would be higher than marginal cost, it follows that an entrant could supply a volume of output slightly higher than some of the incumbent firms at the market price (or slightly lower) and obtain more profit than the incumbent. From lemma 3.10 we know that incumbents obtain zero profits. Therefore, the entrant would get strictly positive profits. This contradicts definition 3.16. □

Lemma 3.12. *In a sustainable configuration with at least two firms, every firm operates where returns to scale are constant.*

Proof. From lemma 3.10 we know that each firm earns zero profits. Accordingly, price equals average cost. Marginal cost equals average cost when returns to scale are constant. As a corollary, since firms are producing where returns to scale are constant, cost is minimized regardless the distribution of the aggregate output among firms. □

Note that the theory of contestable markets does not allow firms following Nash strategies. Baumol (1982) defends the usefulness of the theory in markets where the output of the entrant is small compared to the aggregate production of the industry. In this case, one can justify the ignorance of the price adjustments that incumbents should expect after entry occurs. Martin (2002 supplement) presents extensively and critically other implicit assumptions of the definition of a contestable market such as the absence of sunk costs, of differentiated products, or of transaction costs in financial markets.

According to Spence (1983) the most interesting contribution of this theory lies in the extension to multiproduct firms because provides an analytical technique to study their cost functions. Viewed in perspective, we can conclude that the theory of contestable markets, even if has not succeed in provideng a generalization of the theory of perfectly competitive markets as intended, has proved useful in providing criteria for market regulation policies. The criterion of free entry and exit as a criterion for price flexibility has proved better than previous more rigid criteria[12].

3.6 Variacions 3. Sticky prices. Sweezy's model.

3.6.1 Introduction.

The models we have examined assume that prices are perfectly flexible both upwards and downwards. Nevertheless, it is often observed certain rigidity of prices downwards in oligopolistic environments (see for instance, Purdy, Lindhal and Carter (1950) p.646). Naturally, we should not expect quick price adjustments to small changes in demand and/or costs. Price variations are costly both for firms

[12] Critical appraisals of the structure of contestable markets can be found in Shepherd (1984), Brock (1983), Schwartz (1986) o Martin (1993).

and for consumers. Costs for firms are purely monetary: printing new catalogs, etc. Consumers have to invest in time (and possibly money as well) to gather the new prices. In line with these considerations, empiric studies, although not conclusive, often obtain more evidence of rigidities than of flexibility of prices.

The first attempt to explain this phenomenon goes back to Sweezy (1939). Sweezy's main idea is that an oligopolistic firm when lowering its price should expect its rivals' to react in a similar fashion. But when the firm increases its price, its rivals' should be expected not to react. In other words, Sweezy's construction assumes a more elastic demand for increases than for decreases in prices. Modern treatments of these arguments can be found in Bhaskar (1988), Maskin and Tirole (1988b), or Sen (2004).

3.6.2 Sweezy's model.

Consider a homogeneous product industry where two firms produce with a constant marginal cost technology, k. Market demand is given by $p = A - (q_i + q_j)$.

Assume firms are producing $\widehat{q}_i$ and $\widehat{q}_j$ respectively. Firm i *conjectures* that firm j will continue producing $\widehat{q}_j$ as long as it produces $q_i \leq \widehat{q}_j$ (i.e. price increases). But it also *conjectures* that if it changes its production to $q_i > \widehat{q}_j$ (i.e. price decreases), then firm j will increase its production until level with that of firm i.

Given these conjectures, the only consistent production plans are vectors of the type $\widehat{q}_i = \widehat{q}_j$. If $\widehat{q}_i < \widehat{q}_j$ then, firm j's conjectures say that firm i will increase its production till $\widehat{q}_j$. Mutatis mutandis in the symmetric case $\widehat{q}_i > \widehat{q}_j$. Accordingly, we can restrict the analysis to situations where both firms decide the same production levels $\widehat{q}_i = \widehat{q}_j = \widehat{q}$.

Firm i faces a demand function showing a kink at $q_j = \widehat{q}$.

$$p = \begin{cases} A - q_i - q_j & \text{if } q_i < q_j, \\ A - 2q_i & \text{if } q_i > q_j. \end{cases}$$

Hence marginal revenue function is discontinuous at that point $q_j = \widehat{q}$. In particular,

$$IM_i = \begin{cases} A - q_j - 2q_i & \text{if } q_i < q_j, \\ A - 4q_i & \text{if } q_i > q_j. \end{cases}$$

Figure 3.31 illustrates the situation.

Assume now that firm j produces $q_j = \widehat{q}$. Firm i's problem is

$$\max_{q_i}(A - q_i - y_j(q_i, \widehat{q}) - k)q_i$$

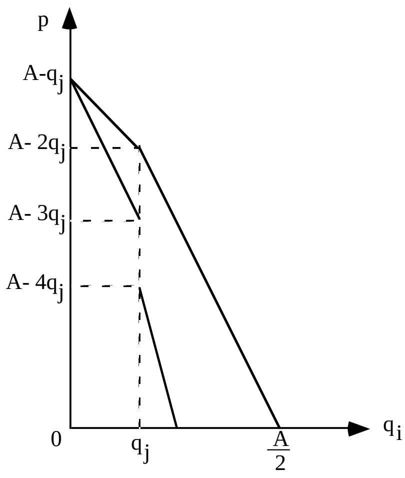

Figure 3.31: Demand and marginal revenue.

where $y_j(q_i, \widehat{q}) = max\{q_i, \widehat{q}\}$. That is, y_j represents firm j's reply. This is to produce $\widehat{q}$ if $q_i \leq \widehat{q}$ and to produce q_i if $q_i > \widehat{q}$. The kink in the demand function originates a very peculiar reaction function. If q_j is large enough (with respect to k), then the equality between marginal revenue and marginal cost appears in the lower segment of the marginal revenue curve; otherwise marginal revenue and marginal cost intersect in the upper part of the marginal revenue curve, as shown in figure 3.32.

Finally, there is a range of values of q_j for which marginal revenue jumps from a point above k to a point below k at the kink. For these values of q_j, firm i's best reply is precisely to adjust its production level to firm j's. Similarly, there is a range of values of k, $k \in [A - 4q_j, A - 3q_j]$ for which marginal cost does not equate marginal revenue (because it is not defined), but the profit maximizing price [quantity] remains constant at the level $A - 2q_j$ $[q_j]$. Formally,

$$q_i^*(q_j) = \begin{cases} 0 & \text{if } q_j \geq A - k, \\ \frac{A-k-q_i}{2} & \text{if } \frac{A-k}{3} \leq q_j \leq A - k, \\ q_j & \text{if } \frac{A-k}{4} \leq q_j \leq \frac{A-k}{3}, \\ \frac{A-k}{4} & \text{if } q_j \leq \frac{A-k}{4}. \end{cases}$$

Figure 3.33 shows this reaction function. Note that $\frac{A-k-q_j}{2} \leq q_j \iff q_j \geq \frac{A-k}{3}$.

In figure 3.33 the continuous line represents firm i's reaction function, while the broken line represents firm j's one. Both curves intersect in the interval $q_i = q_j = \widehat{q} \in [\frac{A-k}{4}, \frac{A-k}{3}]$. Therefore, there is a continuum of equilibria. Note though that in all those equilibria the aggregate production lies in the interval $2\widehat{q} \in [\frac{A-k}{2}, \frac{2(A-k)}{3}]$, that is from the monopoly output to the Cournot output.

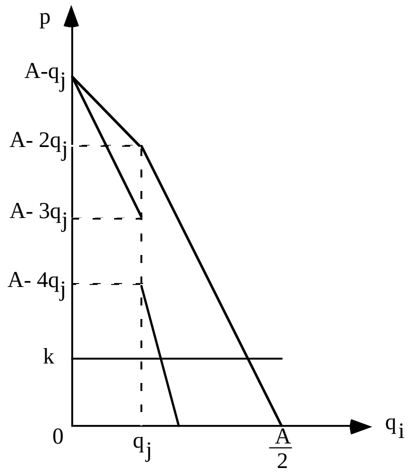

Figure 3.32: Marginal revenue and marginal cost.

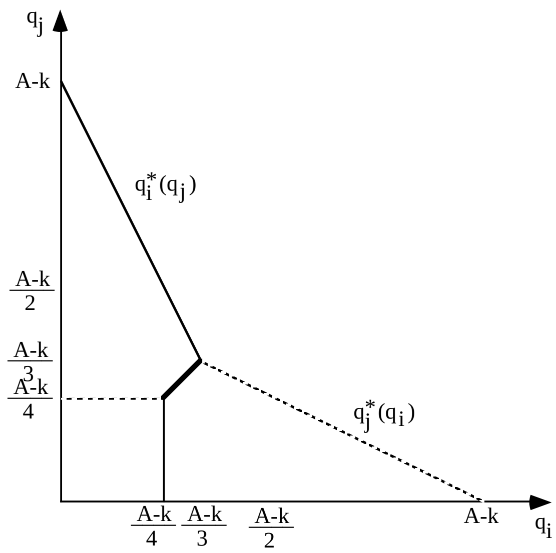

Figure 3.33: Reaction functions.

3.7 Variations 4. Commitment.

The models we have studied so far are static models where all agents take their decisions simultaneously. We have also studied Kreps and Scheinkman's model that is a two-stage model of competition. Two-stage models is an attempt to capture a dynamic behavior with a static model. The fact that agents select their actions in the first stage anticipating the impact on the choice of actions that follows is interpreted as the choice of a long-run decision (first stage) and a short-run decision (second stage). An important use of two-stage models is to capture the idea of commitment. By this we mean that in a situation of strategic interaction, one agent may restrict in a *credible way* its choice set to gain an advantage over a competitor. Stackelberg as far back as 1934 was the first who constructed a model to capture this feature of commitment in oligopoly pricing.

3.7.1 The Stackelberg model.

Stackelberg (1934) departure point is the observation that often in a market there is a firm acting as a leader and several other firms (followers) that, conditional on the behavior of the leader choose their actions. Of course, the leader takes its decision aware of the behavior of the followers. Hence, the strategic interaction appears because the leader's profit maximizing decision is conditional on the reaction of the followers, and the followers' profit maximizing decisions are conditional on the choice of the leader.

Let us assume, for simplicity, a duopolistic industry where demand is given by $P = a - b(q_1 + q_2)$ and firms produce a homogeneous product with a technology described by $C_i(q_i) = c_0 + cq_i,\ i = 1, 2$. Assume also that firm 1 acts as leader and firm 2 as follower[13]. Finally, the strategic variable are quantities. That is Firm 1 chooses its output first. Then, having observed that decision, the follower chooses its output in turn. The commitment appears because the leader once has taken its production decision cannot change it. In formal terms, the strategy of the leader is a number, q_1, while the strategy of the follower is a mapping from the outputs of firm 1 to its set of feasible outputs. Accordingly, a (sub-game perfect) equilibrium is a profit maximizing production plan $(q_1^*, q_2(q_1^*))$.

Given the assumptions on demand and technologies, firm 2 will have a unique best reply to any output of the leader, so that $q_2(q_1^*)$ is simply firm 2's reaction function. This is the solution of

$$\max_{q_2} \Pi_2(q_1, q_2) = \Big(a - b(q_1 + q_2)\Big)q_2 - c_0 - cq_2,$$

[13] Vives (1999, pp. 204-205) provides some comments on the endogenization of the order of moves.

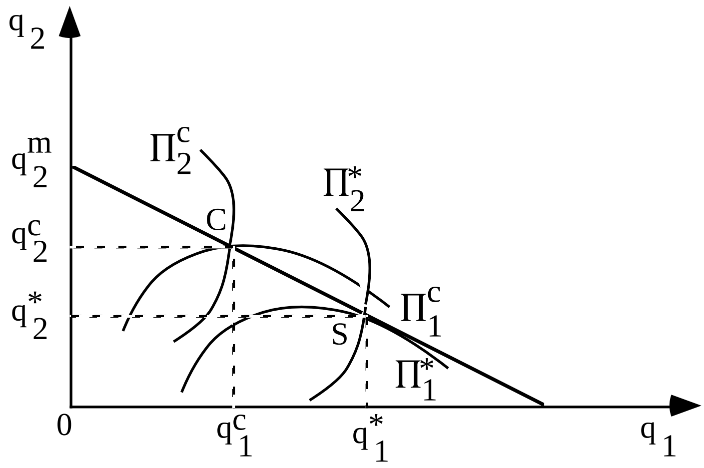

Figure 3.34: Stackelberg equilibrium.

that is,

$$q_2 = \frac{a-c}{2b} - \frac{1}{2}q_1. \tag{3.29}$$

Firm 1's problem is to choose an output level maximizing profits taking anticipating the impact of this decision on the follower. Thus, its problem is

$$\max_{q_1} \Pi_1(q_1, q_2) \text{ s.t. } q_2 = \frac{a-c}{2b} - \frac{1}{2}q_1.$$

Accordingly,

$$q_1^* = \frac{a-c}{2b}. \tag{3.30}$$

Substituting (3.30) in (3.29) we obtain the follower's optimal decision,

$$q_2^* = \frac{a-c}{4b}. \tag{3.31}$$

Figure 3.34 illustrates the decision process just described. The leader chooses a point on a isoprofit curve in the set of points under firm 2's reaction function. The follower simply plugs in the leader's decision q_1^*, to obtain q_2^*.

To complete the description of the industry, from (3.30) and (3.31) we derive the associated market price,

$$P^* = \frac{a+3c}{4},$$

and we can compute the equilibrium profits:

$$\Pi_1^* = \frac{(a-c)^2}{8b} - c_0,$$

$$\Pi_2^* = \frac{(a-c)^2}{16b} - c_0.$$

In order to compare this equilibrium with what would have arisen in a Cournot setting, we present the Cournot equilibrium values:

$$q_1^c = q_2^c = \frac{a-c}{3b}.$$
$$P^c = \frac{a+2c}{3},$$
$$\Pi_1^c = \Pi_2^c = \frac{(a-c)^2}{9b} - c_0.$$

It is easy to check that

$$q_1^* > q_1^c; \quad q_2^* < q_2^c;$$
$$\Pi_1^* > \Pi_1^c; \quad \Pi_2^* < \Pi_2^c;$$
$$Q^* > Q^c; \quad P^* < P^c.$$

Therefore, firm 1 as leader has a strategic advantage ("first-mover advantage) over the follower.

One crucial feature of the analysis is that we are characterizing a subgame perfect equilibrium of the two-stage game. This means that empty (non-credible) threats by the follower are ruled-out[14]. This is so, because the concept of subgame perfect equilibrium requires the follower's strategy to be optimal in front of any decision of the leader q_1, and not only against the equilibrium output q_1^*. In contrast, a Nash equilibrium only requires optimality along the equilibrium path. In our two-stage game, it only imposes production levels for the leader that do not generate loses. In terms of our example, for $C_0 = 0$, any output in $[0, (a-c)/b]$ is sustainable as a Nash equilibrium of the two-stage game.

3.8 Too many models? How to select the "good" one?

So far we have studied several models to understand oligopoly pricing in the context of a homogeneous product industry. Cournot, Bertrand, Stackelberg, Kreps and Scheinkman, Sweezy... All these authors present different conclusions on the outcome of the strategic interaction among firms. The obvious question is then, which of these models is the correct one. Note that these models differ in the behavior that a particular firm conjecture its rivals will follow when choosing its action. Thus it seems reasonable that different models will be adequate for different scenarios.

[14]A statement from the follower towards the leader like "If you do not restrict your output, I will flood the market" will not be credible.

Kreps (1990, pp. 338-340), identifies five different scenarios for which there is a model that best captures its characteristics.

Cournot Two isolated firms decide upon the production levels to send to the market place. Both know the structure of the market and the demand function, but no one knows the decision of the rival. Once production is sent to the market place, a "clearing house" sets the price that clears the market.

In this scenario Cournot conjectures make sense because no firm decides to change its decision when facing the decision of the rival (simply because firms do not have the opportunity to do it).

Stackelberg Consider again the two isolated firms and the clearing house as before. Now one firm decides and sends its production to the clearing house. The rival firm, aware of the situation, decides its production level knowing that the clearing house will set the price equating aggregate demand and supply.

In this scenario the second firm takes the production level of its rival as given and chooses its profit maximizing production according to its reaction function. The first firm, aware of the behavior of its rival, decides its profit maximizing output conditional on the knowledge that its rival will choose a production on its reaction function.

Sweezy Assume now our two isolated firms call, independently, the clearing house to communicate their *intended* production. Then the clearing house calls the firms back. If one of them announces an output level higher than the other, the clearing house allows the latter to produce an amount up to the production of the former.

This scenario is close but not exactly that of Sweezy. In particular, the clearing house has to be aware that if one firm intends to produce more than the other, that one will react increasing its production up to the level of its rival.

Bertrand Imagine now a situation where our two isolated firms communicate, independently, to the clearing house the price at which each is willing to sell its production. The clearing house then evaluates the demand that these prices generate, and call back every firm to communicate their respective production volumes.

Kreps i Scheinkman Finally, consider that our two firms, before producing have to decide, independently an investment in capacity. Firms cannot produce beyond that level. Both firms decide simultaneously their investments. These investment decisions are made public. Next, every firm, independently, calls the clearing house and, as in the Bertrand case, announces the price at which each is willing to sell its production. The clearing house then evaluates the demand that these prices generate. If demand is larger than the aggregate maximum capacity, the clearing house implements some rationing scheme.

These descriptions illustrates the idea that each model makes sense only according to the market mechanism matching demand and supply.

Although different in spirit, as these variety of scenarios illustrates, all these models share some common characteristics. They are static partial equilibrium models, all firms produce a homogeneous product, and most important of all, there is strategic interdependence. This is captured through the so called ' conjectural variations". Conjectural variations is the concept that captures the reaction of one firm to the behavior of its rivals. It appears explicitly (as we have already mentioned in the study of the Cournot model) in the system of first order conditions of the profit maximizing problem of the firms. Different assumptions on the behavior of the firms is thus reflected in the specification of the term $\frac{\partial \sum_{j \neq i} q_j}{\partial q_i}$. Therefore, we can also obtain a better understanding of the different conclusions of the different models presented so far, by viewing them in terms of the underlying assumptions in terms of conjectural variations. This is the object of the next section.

3.9 Variacions 5. Conjectural variations.

The idea of the reaction function[15] captures the strategic interdependence among firms. It is a useful device to envisage changes in the behavior of a firm induced by variations in the behavior of the rivals

Cournot proposed a particular framework of strategic interdependence. Bowley (1924) coined the term *conjectural variation* as a generalization of the concept of strategic interaction. The essential idea is that a firm is aware that its decisions on any strategic variable will affect the decisions of the rival firms. Therefore, the firm wants to incorporate this interaction in its profit maximizing decision process. The particular assumptions of a firm on how his behavior influences the rivals' behavior is summarized in the specification of the conjectural variations.

Recall that in a Cournot model the (implicit) assumption is that any firm takes its decision conditional on the expectation that the rival firms will not vary theirs. This expectations are fulfilled in equilibrium. We can visualize the equilibrium process as a timeless series of actions and reactions. For instance, after firm 1 chooses a certain production level, firm 2 makes its choice taking as given firm 1's

[15]The term *reaction function* is due to Frisch (1933).

decision. In turn, firm 1 observes firm 2's choice and adjusts its decision, then firm 2 reacts and so on, until reaching an equilibrium. To illustrate this process we can think of a timeless game of chess where the players have to plan in advance the sequences of future moves (strategies) according to the expectations (conjectures) on the sequences of moves of the rival. This set of conjectures for all firms is called *conjectural variations*. Extensive treatments are found in Boyer and Moreaux (1983), Bresnahan (1981) or Perry (1982).

3.9.1 Bowley's model.

Let us consider a duopolistic industry[16]. Each firm recognizes the strategic interaction with its rival, and thus takes into account in its profit maximizing decision process. The first order condition of this profit is given by,

$$\frac{\partial \Pi_i}{\partial q_i} + \frac{\partial \Pi_i}{\partial q_j}\frac{dq_j}{dq_i} = 0, \; i \neq j, \tag{3.32}$$

where $\dfrac{dq_j}{dq_i}$ represents firm i's conjecture on the behavior of its rival, firm j, of a marginal variation of its production level, dq_i. In a parallel fashion, firm j also forms its expectations on firm i's behavior when varying its decision q_j. Accordingly, firm j's profit maximizing choice is characterized by,

$$\frac{\partial \Pi_j}{\partial q_j} + \frac{\partial \Pi_j}{\partial q_i}\frac{dq_i}{dq_j} = 0, \; j \neq i. \tag{3.33}$$

The solution of the system (3.32) and (3.33) is a production plan (q_i^*, q_j^*) such that no firm has incentives to unilaterally deviate; in other words, that production plan is a noncooperative (Nash) equilibrium.

To illustrate these ideas, assume a linear demand $p = a - b(q_1 + q_2)$ and a common constant marginal cost c. The first order condition of firm 1 profit maximizing decision process is

$$\frac{\partial \Pi_1}{\partial q_1} = a - c - q_1\big(2b + b\frac{dq_2}{dq_1}\big) - bq_2 = 0.$$

Solving for q_1 we obtain the reaction function,

$$q_1(q_2) = \frac{a - bq_2 - c}{2b + b\Big(\dfrac{dq_2}{dq_1}\Big)}. \tag{3.34}$$

[16] As usual, the argument can easily be generalized to any finite number of firms.

By symmetry, firm 2's reaction function is,

$$q_2(q_1) = \frac{a - bq_1 - c}{2b + b\Big(\dfrac{dq_1}{dq_2}\Big)}. \tag{3.35}$$

Cooperative conjectural variations Assume firms coordinate their output decisions on the monopoly solution, that is, they agree to adjust the aggregate production to the monopoly level by equally adjust their individual outputs. This translates in a conjectural variation $\dfrac{dq_2}{dq_1} = \dfrac{dq_1}{dq_2} = 1$. Substituting these values in (3.34) i (3.35) we obtain,

$$q_1(q_2) = \frac{a - bq_2 - c}{3b}, \tag{3.36}$$

$$q_2(q_1) = \frac{a - bq_1 - c}{3b}. \tag{3.37}$$

To verify that these reaction functions yield the monopoly output, let us first compute the monopoly solution as the maximization of the joint profits. Let $q_1 + q_2 = q_m$.

$$\Pi_m = (a - c)q_m - b(q_m)^2,$$
$$\frac{\partial \Pi_m}{\partial q_m} = a - c - 2bq_m = 0,$$
$$q_m^* = \frac{a - c}{2b}.$$

Going back to our industry, let us now sum the two reactions functions (3.36) and (3.37) to obtain,

$$q_1 + q_2 = \frac{2a - b(q_1 + q_2) - 2c}{3b}.$$

Solving for the aggregate production,

$$q_1 + q_2 = \frac{a - c}{2b}.$$

This is precisely the monopoly production q_m^* obtained above.

Competitive conjectural variations Assume now that every firm conjectures that if it reduces production in one unit, the rival will increase its production in one unit so that the aggregate output remains constant, $\dfrac{dq_2}{dq_1} = \dfrac{dq_1}{dq_2} = -1$.

Substituting these values in (3.34) and (3.35) we obtain,

$$q_1(q_2) = \frac{a - bq_2 - c}{b}, \tag{3.38}$$

$$q_2(q_1) = \frac{a - bq_1 - c}{b}. \tag{3.39}$$

We want to show that these reaction functions generate the competitive production level.

Let us first compute the competitive production by equating market price and marginal cost, assuming $q_1 + q_2 = q_c$.

$$p = c = a - b(q_c),$$

so that,

$$q_c^* = \frac{a - c}{b}.$$

Going back to our industry, let us now sum the reaction functions (3.38) i (3.39) to obtain,

$$q_1 + q_2 = \frac{2a - b(q_1 + q_2) - 2c}{b}.$$

Solving for the aggregate production,

$$q_1 + q_2 = \frac{a - c}{b}.$$

This is precisely the monopoly production q_c^* obtained above.

Cournot conjectural variations An intermediate situation on conjectures is Cournot assumption, where each firm takes the output of the rival as given, $\frac{dq_2}{dq_1} = \frac{dq_1}{dq_2} = 0$. Substituting these values in (3.34) and (3.35) we obtain,

$$q_1(q_2) = \frac{a - bq_2 - c}{2b}, \tag{3.40}$$

$$q_2(q_1) = \frac{a - bq_1 - c}{2b}. \tag{3.41}$$

Summig up the two reactions functions (3.40) and (3.41) we obtain,

$$q_1 + q_2 = \frac{2a - b(q_1 + q_2) - 2c}{2b},$$

and solving for the aggregate production,

$$q_1 + q_2 = \frac{2(a-c)}{3b}.$$

We see that conjectural variations taking values in the range $[-1, 1]$, we can generate the perfect collusion, perfect competition, and Cournot solutions.

Other conjectures Bertrand model contains infinite conjectural variations; Stackelberg model assumes zero conjectural variation for the follower and finite conjectural variation for the leader; Sweezy assumes collusive conjectural variations when output expands, and Cournot conjectural variations when output contracts.

3.9.2 Consistent conjectural variations.

In the previous section we have tried to guess the values of the conjectural variations that are consistent with the equilibrium of some particular models. A different approach consist in trying to find an equilibrium set of conjectural variations. This equilibrium, would have the property that if every firm follows a particular conjectural variations, no firm in the industry would have incentives neither to modify its behavior nor to change their conjectural variations. In other words, this approach tries to identify a Nash equilibrium in conjectural variations. This equilibrium is called a set of *consistent conjectural variations*.

To simplify notation, let us introduce the following,

$$k_1 \equiv \frac{dq_2}{dq_1}; \quad k_2 \equiv \frac{dq_1}{dq_2}$$

Introducing k_1 in (3.34) we obtain,

$$q_1 = \frac{a - bq_2 - c}{2b + bk_1}. \tag{3.42}$$

To find k_2 we differentiate (3.42) with respect to q_2,

$$\frac{dq_1}{dq_2} = k_2 = -\frac{1}{2 + k_1}. \tag{3.43}$$

In a similar way, introducing k_2 in (3.35) we obtain,

$$q_2 = \frac{a - bq_1 - c}{2b + bk_2}. \tag{3.44}$$

To find k_1 we differentiate (3.44) with respect to q_1,

$$\frac{dq_2}{dq_1} = k_1 = -\frac{1}{2+k_2}. \tag{3.45}$$

Substituting (3.45) in (3.43) we obtain,

$$k_2 = -\frac{2+k_2}{3+2k_2}.$$

After some manipulations, the previous equation can be written as $(k_2+1)^2 = 0$, that is $k_2 = -1$. Substituting this value of k_2 in (3.43) we obtain $k_1 = -1$.

Therefore, in our example of linear demand and constant margin al costs, the equilibrium in consistent conjectural variations consists in both firms assuming conjectural variations -1 giving rise to the competitive equilibrium. This results suggests the possibility that the behavior of the firms producing a homogeneous product may be approximated by the competitive behavior although there are few firms in the market.

If we think of this model seriously as a simultaneous decision one, it does not make sense q_i being a function of q_j or viceversa. Such a situation implies that firm i *observes* firm j's decision, and according to its conjectural variation, determines q_i. Therefore, we should think of the conjectural variations as a device to understand the decision process of firms aware of their strategic dependency.

3.9.3 Statics vs dynamics. Marschack and Selten models.

Marschack and Selten (1977, 1978) present two related models. One of them is a static model where the structure of the decision process is explicitly formulated. The second model is dynamic. The link between both models arises from an equivalence between the equilibrium of the static model and a particular equilibrium of the dynamic model.

The model develops in several stages. First, firms simultaneously announce tentative prices; these announcements are made public to all firms; with this information firms can revise their prices. The process of revision follows the following order. The first firm decides whether it wants to revise its price. If it decides not to do it, then the second firm takes its turn, etc. If, on the contrary, the first firm changes its price, then the remaining $n-1$ firms simultaneously decide a new vector of prices. The first firm again has the possibility to revise the price. The process continues until the first firm does not want to change its price anymore. Then the second firm comes into play and the process restart. The final vector of prices obtains when no firm is willing to adjust its price.

This rather complex decision process, allows to overcome the traditional criticism to conjectural variations in static models. Also, conjectural variations are required to be consistent, so that the expected behavior of a firm on its rivals should be correct.

A strategy of a firm i in a static model contains four elements: (a) an initial price, p_i^0; (b) a reply function $\phi_i(p^0, p_j^1)$ stating firm i's price given the present price vector p^0 and a conjecture on firm j's price; (c) the set of price vectors p^0 over which firm i does not want to introduce changes; (d) the changes p_i^0 for those vectors p^0 not in the set defined by (c).

Let us visualize this procedure by assuming that firms have just announced an initial p^0. Firm 1 evaluates the possibility of changing its decision. Assume that it does so and announces a new price p_1^1. Now the present vector of prices is $p^1 = [p_1^1, \phi_2(p^0, p_1^1), \dots, \phi_n(p^0, p_1^1)]$. If given this vector, firm 1 still wants to change its price to p_1^2 the new vector of prices will be $p^2 = [p_1^2, \phi_2(p^1, p_1^2), \dots, \phi_n(p^1, p_1^2)]$. After a finite number of iterations when firm 1 will be happy, firm 2 will start its particular process of adjustment from the price vector that left firm 1 satisfied.

Note that we have described the procedure when a firm i wants to vary its decision after a competitor, firm j, has changed its decision. For completeness, if $p_j^0 = p_j$ then $\phi(p^0, p_j^0) = p_i^0$, that is, if the firm j's new price is the same as in the previous iteration, firm i's reply is to maintain its price; if firm i changes its price, the best reply to itself is the new price, $\phi(p^0, p_i) = p_i$. Accordingly, from an initial price vector p^0, if firm i proposes a change to p_i^1, the next price vector will be,

$$p^1 = \phi(p^0, p_i^1) = [\phi_2(p^0, p_i^1), \phi_2(p^0, p_i^1), \dots, \phi_n(p^0, p_i^1)].$$

If firm i varies k times its decision, $p_i^1, p_i^2, \dots, p_i^k$, after each change the $n-1$ competitors react. We denote the final resulting vector after the k iterations by

$$p^k = \widehat{\phi}(p^0, \{p_i^1, p_i^2, \dots, p_i^k\})$$

that is recursively defined as,

$$p^k = \widehat{\phi}(p^0, \{p_i^1, p_i^2, \dots, p_i^k\}) = \phi\Big[\widehat{\phi}(p^0, \{p_i^1, p_i^2, \dots, p_i^{k-1}\}), p_i^k\Big], (k \geq 2)$$

and for $k = 1$,

$$\widehat{\phi}(p^0, \{p_i^1\}) = \phi(p^0, p_i^1)$$

We refer to the function $\widehat{\phi}_i$ as the *enlarged reply function*.

Definition 3.19. *A non-cooperative equilibrium is a pair (p^0, ϕ) such that,*

(a) $\Pi_i(p^0) \geq \Pi_i[\widehat{\phi}(p^0, \{p_i^1, p_i^2, \dots, p_i^k\})], \ \forall \{p_i^1, p_i^2, \dots, p_i^k\}$

(b) $\Pi_i[\widehat{\phi}(p^0, \{p_i^1, p_i^2, \dots, p_i^k\})] \geq$
$\Pi_i[\widehat{\phi}_i'(p^0, \{p_j^1, p_j^2, \dots, p_j^k\}), \widehat{\phi}_{-i}(p^0, \{p_j^1, p_j^2, \dots, p_j^k\})]$ *for all sequences of price changes* $\{p_j^1, p_j^2, \dots, p_j^k\}, (j \neq i)$, *for all enlarged reply functions* $\widehat{\phi}_i'$, *and for all* $i = 1, 2, \dots, n$.

Condition (a) says that no firm i can improve its profits by deviating from p_i^0, given that the competitor firms will react modifying their prices in accordance with their reaction functions. Condition (b) says that for any sequence of deviations by firm j, given the initial prices p^0 and given the reaction functions of the rivals, firm i's reply function maximizes profits.

In this equilibrium the conjectural variations are consistent because firm i's reaction function, ϕ_i, shows how p_i will be adjusted when any other prices is modified; also ϕ_i is the rule that all competitor firms think firm i will use to decide its price adjustments.

In the intertemporal model, firms after choosing their first price face an adjustment cost for any price change. If a firm i unsuspectedly for its competitors varies its price, there is a period of time between the new price is posted and the competitors adjust their prices. The adjustment cost faced by firm i is sufficiently high to offset the extra profits firm i may obtain in the interim period until the rivals react. Accordingly, a price variation is only profitable if it generates more profits in the long run, once the rivals have reacted.

3.10 Variations 6. Dynamic models.

3.10.1 Introduction.

A common characteristic of the models presented so far is that firms are myopic in the sense that they do not take into account any time horizon. Some instances are one-period models where firms take their decisions simultaneously; other situations present static multi-stage models. In any case, the defining element of any dynamic model, namely the ability to plan ahead (and thus the possibility of transferring present profits to the future) is absent.

The equilibrium market configuration may vary substantially when we allow for repeated interaction due to the presence of durable goods, entry barriers, technological development, etc. Chamberlin (1929) already suggested that in a homogeneous product oligopoly firms are aware of their interaction. Accordingly, the threat of a price war should be effective enough to sustain the monopoly price without need of any explicit cooperation. We postpone the analysis of (tacit) collusion to chapter 5.

An explicit analysis of the dynamics of pricing is difficult. The theory of dynamic games is relatively young and the tools to be used have been developed only recently. Accordingly, there is a certain literature trying to capture dynamic aspects by means of static models. Here we find the contributions of Sweezy, Bowley, Stackelberg, or Edgeworth[17].

[17] See Maskin and Tirole (1988b) for a detailed analysis

There are two alternative ways of explicitly consider the introduction of time in a model. On the one hand, we find models where the dynamic strategies of firms are of Markov type and the objective of the model is to characterize a *Markov perfect equilibrium* (see Maskin and Tirole (1987, 1988a)). On the other hand, we find the so-called *repeated games or supergames*. These replicate a static Cournot (Bertrand) type of game a finite or infinit number of times (see Friedman (1977)).

An overview of dynamic oligopoly models can be found in Fudenberg and Tirole (1986), Kreps and Spence (1984), Shapiro (1989) or Maskin and Tirole (1987, 1988a, 1988b).

3.10.2 Supergames.

As we have mentioned earlier, the problem with static and two-stage models is that they ignore a wide range of strategic possibilities. In repeated games, although one iteration is independent of another, players can condition their present or future behavior to the history of moves. also it allows for the introduction of *punishments* as (credible) threats to affect players' future decisions.

Supergames share three characteristics. First, there are no bounds on the space of strategies. As a consequence, often "folk theorem"-type of results appear. Second, punishments allowing collusive outcomes although may be individually rational, often may not satisfy the incentive compatibility of the set of players. That is after observing a firm deviating from the "collusive strategy" all the other firms in the industry may prefer not to implement the punishment but renegociate a new agreement. Naturally, in such case all incentives to join the coalition disappear. Finally, in repeated games although the set of players remains unaltered along all iterations, intertemporal interaction is not allowed.

To illustrate, let us follow Tirole (1988), and consider a Bertrand game where two firms produce a homogeneous product with the same technology. The firm quoting the lowest price gets all the demand. If several firms quote the lowest price, they evenly share all the demand. Assume now that this game is repeated $T + 1$ times (T finite or infinit). Let us denote firm i's profits in period t as,

$$\pi_i(p_{it}, p_{jt}),\ t = 0, 1, 2, \dots, T$$

Every firm aims at maximizing the present value of the flow of profits,

$$\sum_{t=0}^{T} \delta^t \pi_i(p_{it}, p_{jt})$$

where δ is the discount rate. In every period t both firms simultaneously choose a price. We allow firms to have perfect recall of all the history of past decisions.

Let,

$$H_t = (p_{10}, p_{20}; p_{11}, p_{21}; p_{12}, p_{22}; \ldots; p_{1,t-1}, p_{2,t-1})$$

be the history of prices chosen by both firms up to period t. Then, firm i's strategy depends on H_t.

We want to characterize a perfect equilibrium. That is, for any history H_t in period t, firm i's strategy from t on should maximize the present value of the flow of its future profits conditional on the expectation of firm j's strategy in period t.

If T is finite, the dynamics of the model do not add anything to the static version. By backward induction, in period T the model is equivalent to the static version. Accordingly, the equilibrium strategies in this period will be the same as in the static model, that is, to quote a price equal to the firm's marginal cost. By construction, decisions in period T are not dependent on what happened in the previous period. Therefore, in period $T-1$, everything works as if it would be the last period. Thus, for any H_{T-1}, the equilibrium strategies are again the equality between price and marginal cost. We can repeat this reasoning until the initial period. Summarizing, if the number of iterations is finite, the only equilibrium of the repeated game is simply the iteration in every period of the equilibrium strategies of the static game.

If T is infinite, the outcome of the game is different. First, it is still true that the equilibrium of the finite horizon game is also an equilibrium of the infinite horizon game. To see this, let us consider the following strategy: every firm in every period chooses a price equal to its marginal cost, regardless of the history of past decisions. For every firm, given that the rival quotes a price equal to marginal cost, the best reply is also to choose a price equal to the marginal cost. What is important, is that the infinite horizon model supports many more equilibria. Let p be a price in between the monopoly price and the perfectly competitive one. Consider now the following symmetric strategies: every firm chooses p in period 0. In period t, if both firms have chosen p until $t-1$, then every firm continues choosing p in t; otherwise, firms quote the price equal to marginal cost forever. This is an example of a punishment strategy (see the model of tacit collusion in chapter 5). In accordance with these strategies, if both firms stick to choosing p, they obtain,

$$\frac{1}{2}\Pi(p)(1+\delta+\delta^2+\delta^3+\cdots).$$

If in a certain period s one firm deviates, it obtains at most $\pi(p)$ (because the rival still plays p) in that period, and zero from s on, that is

$$\frac{1}{2}\Pi(p)(1+\delta+\delta^2+\delta^3+\cdots+\delta^{s-1})+\delta^s\pi(p)=\pi(p)\frac{\delta}{2(1-\delta)}.$$

If $\delta \geq (1-\delta)$, or equivalently, $\delta \geq \frac{1}{2}$, deviating from p is not optimal for any firm.

Note that this result is verified for any price in the interval defined by the monopoly price and the perfectly competitive price. In other words, any price can be supported as an equilibrium in the infinitely repeated game. This is an instance of a more general result known as the *folk theorem*.

3.10.3 Multiperiod games.

Let us introduce now the interaction between the different periods of the game. Naturally, as we enlarge the set of strategies (with respect to the repeated games), we should expect and even larger set of equilibria. Most of the analysis has concentrated in the so-called "Markov perfect equilibria". There, firms condition their actions to a reduced subset of state variables rather than in the full history of the game. It is important to realize that the equilibria thus obtained are also equilibria of games with a wider set of strategies. This is so because Markov equilibria are characterized by the set of dynamic reaction functions. These reaction functions are the best replies of every firm to the rivals' decisions, conditioned to the set of state variables. Therefore, if all firm i's rivals behave according to Markov strategies, the best that firm i can do is to behave accordingly.

The interest of using Markov strategies stems from the fact that they allow to model, in a simple fashion, a rational behavior of firms in a dynamic setting. According to Maskin and Tirole (1987, 1988a), they also capture better than the supergames approach, the intuition of the models of industrial organization.

Consider an infinite duopoly à la Cournot. Let us denote by

$$\pi_i(q_{it}, q_{jt}),\ t = 0, 1, 2, \dots$$

firm i's profits in period t when it chooses q_{it} and the rival firm chooses q_{jt}. We assume this function to be twice continuously differentiable, concave in q_{it} and decreasiong in q_{jt},

$$\frac{\partial \pi_i(q_{it}, q_{jt})}{\partial q_{jt}} < 0$$
$$\frac{\partial^2 \pi_i(q_{it}, q_{jt})}{\partial q_{it}^2} < 0$$
$$\frac{\partial^2 \pi_i(q_{it}, q_{jt})}{\partial q_{it} \partial q_{jt}} < 0.$$

Accordingly, reaction functions are well-defined and are negatively sloped. Every firm aims at maximizing the present value of the flow of future profits,

$$\sum_{s=0}^{\infty} \delta^s \pi_i(q_{1,t+s}, q_{2,t+s})$$

where δ denotes the discount factor.

We follow Maskin and Tirole (1987, 1988a) in considering the following decision process: in odd periods, firm 1 decides a production volume that remains fixed until the next odd period, i.e. until $t+2$. In other words, $q_{1,t+1} = q_{1,t}$, if t is odd. Similarly, in even periods firm 2 decides a production volume that remains fixed until the next even period, i.e. $q_{2,t+1} = q_{2,t}$, if t és even.

In every period, we also assume that the relevant state variables are the ones involved in the profit functions. In period $2k+1$, when firm 1 decides, the relevant information is the production of firm 2, $q_{2,2k+1} = q_{2,2k}$. Firm 1's decision is contingent only on $q_{2,2k}$, so that its reaction function is of the type $q_{1,2k+1} = w_1(q_{2,2k})$. Similarly, in even periods when firm 2 chooses its output level, its reaction function is $q_{2,2k+2} = w_2(q_{1,2k+1})$. We call these Markov strategies *dynamic reaction functions*.

The objective of the model is to find a pair (w_1, w_2) that constitutes a perfect equilibrium. That is, for any period t, the dynamic reaction function of a firm must maximize the present value of the discounted flow of future profits given the dynamic reaction function of the rival firm. This pair (w_1, w_2) is called a *Markov perfect equilibrium*.

If (w_1, w_2) is a Markov perfect equilibrium, then for any period $2k+1$ and any firm 2's decision in period $2k, q_{2,2k}$, the choice $q_{1,2k+1} = w_1(q_{2,2k})$ maximizes firm 1's present value of the flow of its future profits given that from this period on, both firms will decide their actions according to (w_1, w_2). Similarly, the parallel condition holds for firm 2. Therefore, these two conditions are sufficient for (w_1, w_2) to be a Markov perfect equilibrium. Hence, it is sufficient to eliminate the profitable deviations of one period. Formally, we say that (w_1, w_2) is a Markov perfect equilibrium if and only if we can identify value functions $[(V_1, Z_1), (V_2, Z_2)]$ such that for any production plan (q_1, q_2),

$$V_i(q_j) = \max_q \Big(\pi_i(q, q_j) + \delta Z_(q)\Big) \tag{3.46}$$

$$w_i(q_j) \text{ maximizes } \Big(\pi_i(q, q_j) + \delta Z_(q)\Big) \tag{3.47}$$

$$Z_i(q_i) = \pi_i(q_i, w_j(q_i)) + \delta V_i(w_j(q_i)),\ i, j = 1, 2;\ i \neq j. \tag{3.48}$$

The function $V_i(q_j)$ represents firm i's present discounted value of the flow of future profits when it is its turn to decide given that in the previous period (and therefore in this period as well) firm j chose q_j, and from now on both firms will take their decisions according to their Markov strategies (w_1, w_2).

The function $Z_i(q_i)$ represents firm i's present discounted value of the flow of future profit when it is committed to an output q_i, it is the rival's turn to choose, and from that moment on both firms will take their decisions according to their Markov strategies (w_1, w_2).

Like in the traditional analysis, since output levels are strategic substitutes, $\dfrac{\partial^2 \pi_i(q_{it}, q_{jt})}{\partial q_{it} \partial q_{jt}} < 0$, reaction functions are decreasing. To find the equilibrium reaction functions we have to solve the system of equations (3.46), (3.47), (3.48).

Assume profit functions are quadratic,

$$\pi_i(q_i, q_j) = q_i(d - q_i - q_j),\ i, j = 1, 2;\ i \neq j.$$

In this case the solution is very simple. Reaction functions are linear and given by $w_1 = w_2 = w$ on $w(q) = a - bq$.

For $\delta = 0$ firms are myopic so that they react according to their respective static reaction functions,

$$w(q) = \frac{d - q}{2}.$$

Hence, $a = \frac{d}{2}$ and $b = \frac{1}{2}$. The stationary state is the Cournot production plan (q_1^c, q_2^c).

For $\delta > 0$, firms not only consider the present profits but also the reaction of the rival in the future. Given that the dynamic reaction functions are decreasing, the intuition tells us that every firm should expand its production beyond the short run level to induce the rival to reduce its production. The stationary production level is given by $q = \frac{a}{1+b}$ and is increasing in δ. The process is dynamically stable, that is, for any initial output level, firms's decisions converge towards the stationary value. Figure 3.35 illustrates the dynamics of this model. The full lines represent the dynamic reaction functions for $\delta \in (0, 1)$; the broken lines represent the static reaction functions. C is the static Cournot equilibrium, and E is the stationary equilibrium.

3.11 Variations 7. Supermodular games.

Amir (1996, 2005)
Vives (1999, 2005a, 2005b)
Topkis (1998)
Milgrom and Roberts (1990)

3.12 Exercises

1. Consider a market with n firms where the demand function $F(Q)$ is downward sloping. No other assumptions are considered. All firms have the same technology described by a increasing and convex cost function. Denote by Q the aggregate output of the n firms, and let $Q_{-j} = \sum_{k \neq j} q_k$.

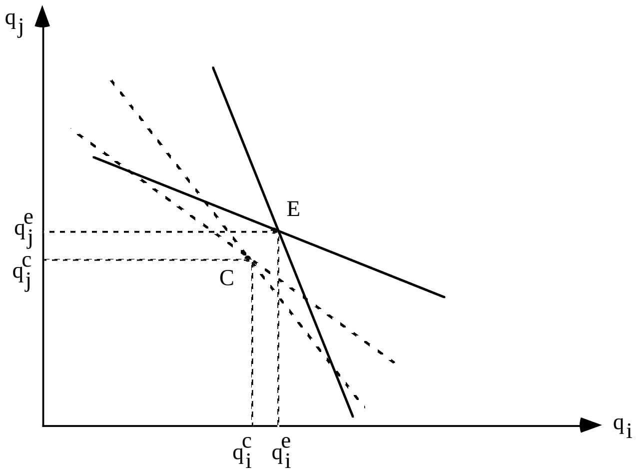

Figure 3.35: Markov perfect equilibrium.

(i) Show that firm j's best response can be written as $b(Q_{-j})$.

(ii) Show that $b(Q_{-j})$ need not be unique (i.e. that is in general a correspondence, not a function).

(iii) Show that if $\hat{Q}_{-j} > Q_{-j},\ q_j \in b(Q_{-j})$, and $\hat{q}_j \in b(\hat{Q}_{-j})$, than $(\hat{q}_j + \hat{Q}_{-j}) \geq (q_j + Q_{-j})$. Deduce from this that $b(\cdot)$ can jump only upwards and that $b^{'}(Q_{-j}) \geq -1$ whenever this derivative is defined.

(iv) Use the result in (iii) to prove that a symmetric pure strategy Nash equilibrium exists in this model.

(v) Show that multiple equilibria are possible.

(vi) Give sufficient conditions for the symmetric equilibrium to be the only equilibrium in pure strategies.

2. Consider an industry with three identical firms. Demand is given by $P = 1 - (q_1 + q_2 + q_3)$. Technology is described by a constant marginal cost equal to zero.

   (i) Compute the Cournot equilibrim;

   (ii) Suppose that two firms merge so that the industry turns into a duopoly. Show that the profits of these firms decrease. Explain;

   (iii) What happens if the three firms merge?.

3. Consider a dupolistic market where firm 1 produces one unit of output using one unit of labor and one unit of raw material. Firm 2 produces one unit of output using two units of labor and one unit of raw material. The unit cost

of labor is w and the unit cost of raw material is r. Demand is given by $P = 1 - (q_1 + q_2)$.

(i) Compute the Cournot equilibrim;

(ii) Use the envelope theorem to show that firm 1's profit is not affected by the price of labor (over some range). Explain.

4. Consider a duopoly Cournot model with linear demand $P = a - b(q_1 + q_2)$ and technologies $C_i(q_i) = c_0 + c_i q_i$. Show that

   (i) a generic isoprofit curve has a maximum;

   (ii) that maximum is a point on the firm's reaction function.

5. Consider a Bertrand duopoly model where firms operate under constant marginal costs c_1 and $c_2,\ c_1 < c_2$. Determine the equilibrium price vector and equilibrium profits for both firms.

6. Consider a n-firm homogeneous product market with downward sloping demand $D(\cdot)$ cutting both axes. Each firm has the cost function $C(q) = F + cq$ if $q > 0$ and $C(0) = 0$. Suppose that a monopoly is strictly viable and that in case of a price tie, a single firm is randomly selected to serve the whole market. Show that the unique Bertrand equilibrium is for all firms to name the least break even monopoly price.

7. Consider a n-firm homogeneous product market with downward sloping demand $D(P) = a - bP$ and costs $C(q) = \dfrac{cq^2}{2}$. Compute the Bertrand equilibrium price when firms split the market in case of a price tie.

8. Consider a variation of Bertrand symmetric model in which prices must be named in some discrete units of size Δ.

   (i) Show that both firms naming prices equal to the smallest multiple of Δ that is strictly greater than c is a pure strategy equilibrium.

   (ii) Argue that as $\Delta \to 0$, the equilibrium converges to both firms charging prices equal to c.

   (iii) Assume now that marginal cost are constant but different, namely $c_1 < c_2$. If prices are named in discrete units as in (ii), what a the pure strategy equilibria of the game? As the grid becomes finer, what is the limit of these equilibria?

9. Consider a two-firm Cournot model with constant but different marginal costs $c_1 > c_2$. Assume also that the inverse demand function is $p(q) = a - bq$ with $a > c_1$.

   (i) Derive the Nash equilibrium of the model. Under what conditions does it involve only one firm producing; Which will this be?

   (ii) When the equilibrium involves both firms producing, how do equilibrium outputs and profits vary when firm 1's cost changes?

10. Consider two strictly concave and differentiable profit functions $\pi_j(q_j, q_k),\ j = 1, 2$ defined on $q_j \in [0, q]$.

    (i) Give sufficient conditions for the best response functions $b_j(q_j)$ to be increasing or decreasing.

    (ii) Argue that in a Cournot model a downward-sloping reaction function is the "normal" case.

11. Consider a market with demand $P = 1 - q$ where three firms produce a homogeneous product with a zero marginal cost technology, up to a production level K, and infinite marginal cost beyond. Compute the Nash equilibrium of the game in pure strategies.

12. Consider a market with two firms producing a homogeneous product. Market demand is given by $q = D(P)$ and assumed well-defined, differentiable, bounded and strictly decreasing. Both firms have the same technology described by a cost function cq. Let $R_i(P_1, P_2, q)$ denote the residual demand faced by firm i. It is defined as,

$$R_i(P_1, P_2, q) = \begin{cases} \max\left\{0, D(P_i) - q_j\left[\lambda, (1-\lambda)\frac{D(P_i)}{D(P_j)}\right]\right\} \text{ if } P_i > P_j, \\ \max\left\{\frac{D(P_i)}{2}, D(P_i) - q_j\right\} \text{ if } P_i = P_j, \\ D(P_i) \text{ if } P_i < P_j, \end{cases}$$

where $\lambda \in [0, 1]$.

   (a) Provide an interpretation of the elements of $R_i(\cdot)$

   (b) Assume firms decide simultaneously their prices and output volumes. Solve for the pure strategy Nash equilibrium of the game.

   (b) Assume firms decide sequentially first their prices (stage 1) and then their output volumes (stage 2). Solve for subgame perfect Nash equilibrium of the game in pure strategies.

13. Consider a Stackelberg duopoly with market demand given by $p = a - b(q_1 + q_2)$. Firms produce with a technology $C_i(q_i) = c_0 + cq_i,\ i = 1, 2$. Let firm 1 be the leader. Compute the Nash equilibria of the game.

14. Consider a duopoly à la Kreps-Scheinkman with zero marginal costs and c_0 unit capacity cost. Assume rationing follows the efficient rule and the following demand function:

$$D(p) = \begin{cases} 1 & \text{if } p \leq 1 \\ 0 & \text{if } p > 1 \end{cases}$$

Compute the reduced-form profit functions and the equilibrium strategies. Show that the Cournot outcome is the solution of the two-stage game.

1. Considerem un mercat amb funció de demanda $q = 200 - 2p$ on operen una empresa dominant i una "franja competitiva" composta per petites empreses. Les empreses petites consideren el preu de l'empresa dominant com donat i ofereixen una quantitat agregada $S = p - 70, (p > 70)$, on p és el preu fixat per l'empresa dominant. Determinar la solució òptima de l'empresa dominant quan el seu cost marginal és costant i igual a (i) $c = 70$, (ii) $c = 45$, (iii) $c = 20$.

2. Considerem un mercat de producte homogeni amb funció de demanda $P = 150 - 4Q$. Hi ha dues empreses produint amb costos marginals constants i iguals a 40.

   (a) determinar els valors d'equilibri de Cournot (preus, quantitats, beneficis);

   (b) calcular la pèrdua d'eficiència com percentatge de la pèrdua d'eficiència en situació de monopoli;

   (c) refer l'exercici suposant que hi ha vuit empreses en lloc de dues.

3. Considerem un duopoli on la demanda és $Q = 10 - \frac{1}{2}P$.

   (a) determinar l'equilibri de Cournot quan ambdues empreses tenen la mateixa tecnologia $C_i(q_i) = 10 + q_i(q_i + 1)\ i = 1, 2$;

   (b) Quin és l'equilibri si les funcions de cost són $C_1 = 10 + 2q_1$ i $C_2 = 10 + \frac{3}{2}q_2$?

4. Considerem una indústria composta per vuit empreses. Cinc d'aquestes empreses utilitzen una tecnologia antiga amb productivitat de 0.25 unitats per hora de treball. Les restants tres empreses utilitzen una tecnologia moderna amb productivitat de 0.45 unitats per hora de treball. La demanda del mercat és $Q = 500000 - 10P$ i el salari per hora és $w = 500$.

(a) determinar l'equilibri de Cournot;

(b) verificar la fórmula $\Im_a = \frac{1}{\varepsilon} C_H$;

(c) calcular el valor màxim que una empresa estaria disposada a pagar per adoptar la tecnologia moderna, suposant que la resta d'empreses no varien la tecnologia;

(d) quin és l'impacte sobre les quotes de mercat d'un augment del salari en un 50%?;

(e) recalcular el valor màxim que una empresa estaria disposada a pagar per adoptar la tecnologia moderna després de l'augment del salari en un 50%.

5. Demostrar que en un duopoli de Cournot (amb costos marginals constants), les quantitats i beneficis d'equilibri de cada empresa són funcions decreixents en el cost marginal de la pròpia empresa i creixents en el cost marginal de l'empresa rival.

6. Considerem un duopoli de Cournot amb demanda lineal $P = 1 - Q$ i costos marginals constants zero per ambdues empreses.

   (a) calcular l'equilibri de Cournot;

   (b) suposem ara que una de les empreses és pública i té com a objectiu maximitzar l'excedent total. Com varien els preus, quantitats, beneficis i excedent total respecte a l'equilibri de Cournot?;

   (c) refer l'apartat (b) suposant que els costos marginals són $c_2 < c_1$;

   (d) refer l'apartat (c) suposant que ambdues empreses són públiques;

   (e) comentar: "l'anàlisi dels beneficis no és suficient per comparar les empreses públiques i privades".

7. Considerem un mercat duopolístic amb funció inversa de demanda $p = 100 - 0.1Q$ i funcions de costos $C_1(q_1) = 6000 + 16q_1$ i $C_2(q_2) = 9000 + 10q_2$.

   (a) calcular l'equilibri de Cournot;

   (b) calcular la frontera de possibilitats de beneficis i comprovar que l'equilibri de Cournot no és òptim de Pareto.

8. Considerem un mercat amb funció de demanda $q = 200 - 2p$ constituït per una empresa dominant i 10 empreses petites que formen una franja competitiva. Les empreses petites prenen com donat el preu de l'empresa dominant i ofereixen una quantitat agregada $S = p - 70 (p > 70)$, on p és el

preu fixat per l'empresa dominant. La demanda residual és satisfeta per l'empresa dominant. Determinar la solució òptima per l'empresa dominant quan opera amb un cost marginal constant i igual a (i) $c = 70$; (ii) $c = 45$ i (iii) $c = 20$.

9. Considerem un duopoli de Cournot amb funció de demanda $Q = 500-50p$. L'empresa 1 opera amb un cost marginal constant $c_1 = 8$; l'empresa 2 té un cost marginal constant $c_2 = 6$ i la seva capacitat de producció limitada a 25 unitats. Calcular els valors d'equilibri.

10. Considerem un mercat de producte homogeni en el que operen quatre empreses. La quota de mercat de la primera d'elles és el doble de la mitja de les quotes de mercat de les tres altres empreses. Deduir els valors numèrics dels paràmetres del models de Cournot i Stackelberg consistents amb aquesta distribució de quotes de mercat.

11. Considerem un duopoli à la Bertrand on les empreses tenen costos marginals constants $c_i;\ c_1 < c_2$. Demostrar

    (a) que ambdues empreses fixen el preu $p = c_2$;

    (b) que l'empresa 1 obté un benefici de $(c_2 - c_1)D(c_2)$, i que l'empresa 2 no obté beneficis si $c_2 \leq p^m(c_1)$ on $p^m(c_1)$ maximitza $(p - c_1)D(p)$; (si $c_2 > p^m(c_1)$, l'empresa 1 fixa $p^m(c_1)$).

12. En un mercat hi ha dues empreses que operen amb una tecnologia $C(q) = q^2/2$. La demanda és $p = 1 - (q_1 + q_2)$.

    (a) calcular l'equilibri de Cournot;

    (b) suposem que l'empresa 1 té l'oportunitat de vendre x_1 unitats del bé en un altre mercat; per tant el cost de l'empresa és $(q_1 + x_1)^2/2$. La demanda en el segon mercat és $p = a-x$. Considerar el joc de Cournot en el que l'empresa 1 escull q_1 i x_1 i, simultàniament, l'empresa 2 escull q_2. Demostrar que $q_1 = (2-a)/7$ i $q_2 = (5+a)/21$ en l'interval rellevant de valors de a. Demostrar (utilitzar el teorema de l'envolvent) que per $a = 1/2$ un petit increment de a perjudica a l'empresa 1. Interpretar el resultat.

13. Dues empreses produeixen un bé homogeni. Sigui p el preu del producte, q_i el nivell de producció de l'empresa $i,\ i = 1, 2$, i $Q \equiv q_1 + q_2$. La demanda d'aquest producte és $p = \alpha - Q$. El cost marginal de l'empresa i és c_i, on $\alpha > c_2 > c_1 > 0$.

    (a) Trobar l'equilibri de Cournot.

(b) Trobar l'equilibri de Stackelberg quan l'empresa 1 és líder.

(c) Trobar l'equilibri de Stackelberg quan l'empresa 2 és líder. Hi ha alguna diferència en la distribució dels volums de producció entre aquest cas i el cas del apartat (b)?. Explicar.

(d) Trobar l'equilibri de Bertrand.

14. En una indústria hi ha N empreses produint un producte homogeni. sigui q_i el volum de producció de l'empresa $i, \ i = 1, 2,$ i $Q \equiv \sum_{i=1}^{N} q_i$. La demanda de mercat és $p = 100 - Q$. La funció de cost total de l'empresa i és

$$TC_i(q_i) = \begin{cases} F + (q_i)^2 & \text{if } q_i > 0 \\ 0 & \text{if } q_i = 0 \end{cases}$$

(a) Suposem que N és prou petit perquè les empreses obtinguin beneficis extraordinaris. Calcular els volums de producció i de beneficis en l'equilibri de Cournot.

(b) Suposem ara, que les empreses poden entrar i sortir de la indústria. Trobar el número d'empreses d'equilibri en la indústria com a funció de F.

15. Considerem un mercat amb funció de demanda $p = 120 - Q$. Suposem que hi ha tres empreses que decideixen els seus volums de producció de forma seqüencial: l'empresa 1 decideix q_1 en el període 1, l'empresa 2 decideix q_2 en el període 2, i l'empresa 3 decideix q_3 en el període 3. Una vegada les tres empreses han decidit els seus volums de producció, venen el volum de producció agregat en el mercat i obtenen beneficis. Trobar els volums de producció d'equilibri.

16. Considerem el mercat duopolístic del problema 7.

(a) derivar l'equilibri de variacions conjecturals, i verificar per quins valors de les variacions conjecturals obtenim l'equilibri de Cournot.

(b) derivar la condició de consistència de les variacions conjecturals.