CONSISTENT VERSUS NON-CONSISTENT CONJECTURES IN DUOPOLY THEORY: SOME EXAMPLES*

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Conjectural variations are at the root of duopoly theory. Cournot [5] was the first formally to study the problem of competition arising when only two firms are selling on a market for an homogenous good. In deriving his equilibrium concept for such a market situation, Cournot made the hypothesis that firm $i$ would determine its production level $q_i$ considering as given the production level of its competitor, firm $j$. Hence Cournot duopolists are said to have zero conjectural variations: each one is conjecturing that the competitor will not react to a change in one’s production level. A Cournot equilibrium is reached when, considering both the production level of the competitor and its own production level and given the zero conjectural variations, none of the firms wants to change its own production level. As is well known, different solutions can be reached if non-zero conjectural variations are assumed. Kamien [9] suggests that any market configuration $(q_i, q_j)$ can be accounted for by assuming appropriate conjectural variation coefficients for the duopolists. A major flaw of these models is that in all but very limited cases the conjectures one assumes to begin with are not validated at the equilibrium. Hence the models are inconsistent: the firms are assumed to hold conjectures which turn out to be different from the optimal reaction of the competitor, that is the reaction which restabilizes the competitor in a profit maximizing solution once the change in the firm’s production level is made. A growing literature has developed recently to analyse the implications for duopoly theory of requiring that the conjectures held by the firms be consistent or rational. One may mention [1], [2], [3], [4], [10], [11], [12], and [15] for the partial equilibrium approach and [6], [7], [8], and [13] for a general equilibrium approach.

In this paper, we want to illustrate through examples that these consistency conditions do not restrict the wide diversity of a priori possible equilibria. As Boyer and Moreaux [1] have shown, extending Laitner’s [11] previous results, practically any situation can be understood as a locally consistent conjectural equilibrium. Here, we work out three well known cases to demonstrate their rationality at equilibrium: Cournot duopolists as well as

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Stackelberg duopolists are thus seen to be correct when adopting apparently inconsistent conjectural variations! We also give an algorithm to compute, given any sustainable market situation \((q_1, q_2)\), conjectural variations functions which make this market situation a locally rational conjectural equilibrium. The paper is organized as follows. In section I we first indicate why the traditional formulation of the conjectural variation duopoly models seems implausibly restrictive and we proceed to reformulate those models in a more acceptable framework. In section II we show in the simple framework of a linear model how some classical situations (Cournot \([5]\), Stackelberg \([14]\)) can appear as rational conjectural equilibria. In section III, the general algorithm is given. In the conclusion we discuss the implications of the results and give suggestions for future work.

I. CONSISTENT CONJECTURES

Let \(p(Q)\) be the inverse demand function on some market, with two firms, \(j = 1, 2\), whose cost functions are \(C_j(q_j)\). The profits functions are \(\pi_j(q_1, q_2) = P(q_1 + q_2)q_j - C_j(q_j)\). The first order conditions of profit maximization are then:

\[
\frac{d\pi_j}{dq_j} = P(q_1 + q_2) + q_j P'(q_1 + q_2) \left[ 1 + \left( \frac{dq_i}{dq_j} \right)^C \right] - C_j = 0 \quad j = 1, 2
\]

\(i \neq j\)

where \(\left( \frac{dq_i}{dq_j} \right)^C\) is the conjectural variation held by firm \(j\).

Provided the higher order conditions are fulfilled, any pair \((q_1^*, q_2^*)\), solution of this system, is a local equilibrium. In these first order conditions the term \(\left( \frac{dq_i}{dq_j} \right)^C\) is the reaction of firm \(i\), conjectured by firm \(j\), to a variation of firm \(j\)'s own production level. In the usual duopoly models these terms are assumed constant. Clearly, it is not a very plausible assumption but let us proceed with it and suppose that \(\left( \frac{dq_i}{dq_j} \right)^C = \alpha_j\), and \(\left( \frac{dq_j}{dq_i} \right)^C = \alpha_i\). One may wonder: what would firm \(j\) observe if it were to change slightly its production level from its equilibrium level? Firm \(i\) would react in order to reach a point of maximum profit given the new production level of firm \(j\). This reaction must then satisfy:

\[
\frac{\partial}{\partial q_i} \left( \frac{dq_i}{dq_i} \right) dq_i + \frac{\partial}{\partial q_j} \left( \frac{dq_i}{dq_i} \right) dq_j = 0,
\]

so that:

\[
\frac{dq_i}{dq_j} = -\frac{\partial}{\partial q_j} \left( \frac{dq_i}{dq_i} \right)
\]

\[
\frac{dq_i}{dq_i} = -\frac{\partial}{\partial q_i} \left( \frac{dq_i}{dq_i} \right)
\]
all the functions being evaluated at \((q_1^*, q_2^*)\); if the conjectural variations are constant, it means that:

\[
\frac{\partial}{\partial q_i} \left( \frac{dn_i}{dq_i} \right) = P' + P'[1 + \alpha_i] + q_i^*P''[1 + \alpha_i] - C_i'
\]

\[
\frac{\partial}{\partial q_j} \left( \frac{dn_i}{dq_i} \right) = P' + q_i^*P''[1 + \alpha_i]
\]

hence:

\[
dq_i \quad dq_j = -\frac{P' + q_i^*P''[1 + \alpha_i]}{P'[2 + \alpha_i] + q_i^*P''[1 + \alpha_i] - C_i'}
\]

The consistency of conjectures requires that the reaction of firm \(i\), conjectured by firm \(j\), be equal to the reaction which firm \(i\) will want to have; i.e., it requires that \((dq_i/dq_j)^C = (dq_i/dq_j)\); hence: from (4),

\[
\alpha_j = -\frac{P' + q_i^*P''[1 + \alpha_i]}{P'[2 + \alpha_i] + q_i^*P''[1 + \alpha_i] - C_i'} \quad i \neq j
\]

and symmetrically for the reaction of firm \(j\) conjectured by firm \(i\). We have then two conditions for the two parameters \(\alpha_i\) and \(\alpha_j\). Kamien and Schwartz [10], Perry [12] and Ulph [15] have shown that in particular situations these conditions may be very stringent.

But one may rightly ask why the conjectural variation should be a constant? It seems that the conjectures formed by the firms should take into account and depend on the characteristics of the situation including the production levels of the two firms, \(q_i\) and \(q_j\). A priori many functional forms could be specified, but the simpler one is the linear form:

\[
(dq_i/dq_j)^C = \alpha_j + \beta_j q_j + \gamma_j q_i \quad j = 1, 2 \quad i \neq j
\]

It is difficult a priori to accept that the conjectural variation formed by firm \(i\) would be constant or independent of either its own production level \(q_i\) or its competitor’s production level \(q_j\). Since the linear form (6) is the simplest possible form, we propose to use it and see what it implies.

Assuming such a conjectural variation function, the first order conditions become:

\[
\frac{dn_j}{dq_j} = P(q_1 + q_2) + q_j P'(q_1 + q_2)
\]

\[
\times [1 + \alpha_j + \beta_j q_j + \gamma_j q_i] - C_j' = 0 \quad j = 1, 2 \quad i \neq j
\]
Let \((q_1^*, q_2^*)\) be a solution of these two equations. What is firm \(i\)'s reaction to a slight departure of firm \(j\) from \(q_j^*\)? Differentiating firm \(i\)'s first order condition gives:

\[
\frac{\partial}{\partial q_i} \left( \frac{d r_i}{d q_i} \right) = P^* \cdot [2 + \alpha_i + 2\beta_i q_i^* + \gamma_i q_j^*] \\
+ q_i^* \cdot P'' \cdot [1 + \alpha_i + \beta_i q_i^* + \gamma_i q_j^*] - C_i^* \\
= Z_i(\alpha_i, \beta_i, \gamma_i, q_i^*, q_j^*)
\]

\[
\frac{\partial}{\partial q_j} \left( \frac{d r_i}{d q_i} \right) = P^* + q_i^* [P'' [1 + \alpha_i + \beta_j q_i^* + \gamma_j q_j^*] + P'\gamma_i] \\
= Y_i(\alpha_i, \beta_i, \gamma_i, q_i^*, q_j^*)
\]

Considering (3), conjectures (6) will be consistent if the following equation holds:

\[
(8) \quad \frac{d q_i}{d q_j} = \alpha_j + \beta_j q_j^* + \gamma_j q_i^* = - \frac{Z_i(\alpha_i, \beta_i, \gamma_i, q_i^*, q_j^*)}{Y_i(\alpha_i, \beta_i, \gamma_i, q_i^*, q_j^*)} \quad j = 1, 2
\]

Let us now consider the following problem: given a pair \((q_1, q_2) > 0\), is it possible to find linear conjectural variation functions such that this pair could be a consistent conjectural equilibrium? Clearly what is to be found are the six values \(\alpha_i, \beta_i, \gamma_i, \alpha_j, \beta_j, \gamma_j\) of (6) satisfying a non linear system of four equations, the two first order equations (7) and the two consistent conjectures equations (8), and two inequalities stemming from the second order conditions. Once \((q_1, q_2)\) is given, all the elements of this system are taking given values except for \(\alpha_1, \beta_1, \gamma_1, \alpha_2, \gamma_2, \beta_2\). Boyer and Moreaux [1], and [2] have shown that such values exist, under very weak assumptions. In the next section we give three examples of classical duopoly models which are generally qualified as inconsistent but which can be seen to be locally consistent with proper linear conjectural variations functions. In the following section, we present an algorithm which can be used to generate any such case.

II. THREE EXAMPLES OF CONSISTENT CONJECTURAL EQUILIBRIA

Consider the following linear model where the inverse demand function is \(P(Q) = \max[a - bQ, 0]\), \(a, b > 0\), and the cost functions of the two firms \(C_j(q_j) = c q_j, j = 1, 2, c > 0, a - c > 0\). It is well known that for this case, the competitive total output is \((a - c)/b\), and the collusive or monopoly total is \((a - c)/2b\). In this section we show how some classical duopoly configurations can be obtained as rational conjectural equilibria.
Example 1: *The Cournot solution*

With the above specification of the demand and cost functions, one can easily obtain that the Cournot solution will be

\[ q_1 = q_2 = \frac{a - c}{3b} \]

giving a total output of

\[ \frac{2}{3} \left( \frac{a - c}{b} \right). \]

This is the solution of Cournot’s reaction functions obtained by assuming zero conjectural variations in (1) above. We obtain

\[ a - 2bq_j - bq_i - c = 0, \quad i, j = 1, 2 \quad \text{and} \quad i \neq j \]

Hence the slope of this reaction function of firm \( j \) is

\[ \frac{dq_j}{dq_i} = -\frac{1}{2} \]

showing that the conjectural variation held by firm \( i \)

\[ \left( \frac{dq_j}{dq_i} \right)_c = 0 \]

is inconsistent.

Now consider the following.

Suppose the conjectural variation functions have the following form:

(10.1) Firm #1 \( \left( \frac{dq_2}{dq_1} \right)_c = \frac{2}{3} + \left( \frac{b}{a - c} \right)q_1 + \left( \frac{3b}{c - a} \right)q_2 \)

(10.2) Firm #2 \( \left( \frac{dq_1}{dq_2} \right)_c = \frac{1}{3} + \left( \frac{2b}{a - c} \right)q_2 + \left( \frac{3b}{c - a} \right)q_1 \)

The conjectural variation functions are not symmetric, yet there exists a symmetric equilibrium. The first order conditions for profit maximisation are

\[ \frac{d \pi_1}{dq_1} = \frac{a - c}{b} - \frac{8}{3} q_1 - \frac{b}{a - c} q_1^2 - q_2 + \frac{3b}{a - c} q_1 q_2 = 0 \]

\[ \frac{d \pi_2}{dq_2} = \frac{a - c}{b} - \frac{7}{3} q_2 - \frac{2b}{a - c} q_2^2 - q_1 + \frac{3b}{a - c} q_1 q_2 = 0 \]

It can be easily checked that these equations have a symmetric solution

\[ (q^*, q^*) = \left( \frac{a - c}{3b}, \frac{a - c}{3b} \right). \]
Furthermore the second order conditions for a conjectured maximum are satisfied. Indeed the second order condition is

\[ \frac{d}{dq_j} \left( \frac{d^2J}{dq_i dq_j} \right) = 2P \left[ 1 + \left( \frac{dq_i}{dq_j} \right)^C \right] + q_j P^r \left[ 1 + \left( \frac{dq_i}{dq_j} \right)^C \right]^2 \]

\[ + q_j^* P^r \left[ \frac{\partial}{\partial q_i} \left( \frac{dq_i}{dq_j} \right)^C \cdot \left( \frac{dq_i}{dq_j} \right)^C + \frac{\partial}{\partial q_j} \left( \frac{dq_i}{dq_j} \right)^C \right] - C_j^r < 0 \]

\[ j = 1, 2 \]

\[ i \neq j \]

so that at \((q_1^*, q_2^*)\) we get after routine calculations:

\[ \frac{d}{dq_1} \left( \frac{d^2J}{dq_1 dq_1} \right) = -\frac{7}{3} < 0 \quad \text{and} \quad \frac{d}{dq_2} \left( \frac{d^2J}{dq_2 dq_2} \right) = -\frac{8}{3} < 0 \]

Hence \((q_1^*, q_2^*)\) is an equilibrium, solving the two equations (11). To verify that this equilibrium is consistent, one must check that conditions (8) are satisfied. Indeed, one finds that at

\[ \left( q_1^*, q_2^* \right) = \left( \frac{a - c}{3b}, \frac{a - c}{3b} \right) \]

the conjectural variation functions (10) take the values

\[ \left( \frac{\partial q_2}{\partial q_1} \right)^C = \left( \frac{\partial q_1}{\partial q_2} \right)^C = 0 \]

and differentiating implicitly the reaction functions (11) one gets for firm 2

\[ \left[ -\frac{7}{3} - \frac{4b}{a - c} q_2 + \frac{3b}{a - c} q_1 \right] dq_2 - \left[ 1 - \frac{3b}{a - c} q_2 \right] dq_1 = 0 \]

hence at \((q_1^*, q_2^*)\) the reaction of firm #2 to a slight modification of the production level of firm #1 is given by:

\[ \frac{dq_2}{dq_1} = \frac{1 - 1}{1 - (11/3)} = 0 \]

which is precisely the value of conjectural variation function used by firm #1; and differentiating the first order condition of firm #1, we have:

\[ \left[ -\frac{8}{3} - \frac{2b}{a - c} q_1 + \frac{3b}{a - c} q_2 \right] dq_2 - \left[ 1 - \frac{3b}{a - c} \right] dq_1 = 0 \]

and at \((q_1^*, q_2^*)\) the reaction of firm #1 to a slight modification of the output level of firm #2 is given by:

\[ \frac{dq_1}{dq_2} = \frac{1 - 1}{1 - (10/3)} = 0 \]
which is precisely the value of the conjectural variation function used by firm \#2. So the equilibrium appears as a rational conjectural equilibrium and the quantities are those of the Cournot equilibrium. Where is the problem? In the traditional Cournot model the conjectural variation functions are:

\[
\left( \frac{dq_i}{dq_j} \right)^c = 0 \quad Vq_j, Vq_i, \quad j = 1, 2; \quad i \neq j
\]

These functions do not generate a rational conjectural equilibrium because, as we saw above, with such conjectural variation functions, the effective reaction of firm \#i to the production of firm \#j gives:

\[
\frac{dq_i}{dq_j} = -\frac{1}{2} \quad j = 1, 2; \quad i \neq j
\]

It must be pointed out that other conjectural variations functions than those used here can generate a rational equilibrium with the Cournotian production levels. Symmetric functions are immediate candidates, as for example

\[
(12) \quad \left( \frac{dq_i}{dq_j} \right)^c = \frac{2}{3} + \left( \frac{b}{a-c} \right)q_j + \left( \frac{3b}{c-a} \right)q_i \quad j = 1, 2; \quad i \neq j
\]

But many other functions, symmetric or not, could imply the same result. We give in section III an algorithm to compute the coefficients of the conjectural variation functions which permit one to obtain local rational conjectural equilibria. As the reader may then verify, infinitely many conjectural variation functions could give rise to a rational conjectural equilibrium with the Cournot output levels.

**Example 2: The weak Stackelberg solution**

The Stackelberg leader-follower model of duopoly is based on asymmetric behavior of the two firms. The leader is assumed to know the Cournotian reaction function of the follower. The latter behaves as a Cournot duopolist, that is with a zero conjectural variation. If, rather than assuming that the leader knows the follower’s reaction function, we assume that the leader knows the follower’s reaction slope only, which is \(-\frac{1}{2}\) here as we noted before, then we have the weak Stackelberg solution. It is the same as the Stackelberg solution but allows us to define a reaction function for the leader as well. With the demand and cost functions specified above, one derives the weak Stackelberg solution with firm 1 as the leader as

\[
q_1^* = \frac{a-c}{2b}, \quad q_2^* = \frac{a-c}{4b}
\]

giving a total output of \(\frac{3}{4} \left( \frac{a-c}{b} \right)\).
The slope of the follower’s reaction function is indeed \(-\frac{1}{2}\) so that the leader’s conjecture is consistent but the slope of the leader’s reaction function—in the weak Stackelberg model—is \((-\frac{1}{2})\) and therefore the follower’s conjecture is inconsistent.

Now consider the following. Assume that the conjectural variation functions are

\[
\text{Firm} \ #1 \quad \left( \frac{dq_2}{dq_1} \right)^C = \frac{c - a}{2b} + q_1 + \frac{2b}{c - a} q_2
\]

(13)

\[
\text{Firm} \ #2 \quad \left( \frac{dq_1}{dq_2} \right)^C = \frac{c - a}{2b} + q_2 + \frac{1}{2} q_1
\]

The first order conditions (1) now are:

\[
\frac{d\pi_1}{dq_1} = \frac{a - c}{b} - \left(2 + \frac{c - a}{2b}\right)q_1 - q_2 - q_1^2 + \frac{2b}{a - c} q_1 q_2 = 0
\]

(14)

\[
\frac{d\pi_2}{dq_2} = \frac{a - c}{b} - \left(2 + \frac{c - a}{2b}\right)q_2 - q_1 - q_2^2 - \frac{1}{2} q_1 q_2 = 0
\]

A solution is \((q^*_1, q^*_2)\) = \(\left(\frac{a - c}{2b}, \frac{a - c}{4b}\right)\), that is the Stackelberg quantities.

The second order conditions are satisfied at \((q^*_1, q^*_2)\), since then

\[
\frac{d}{dq_1} \left( \frac{d\pi_1}{dq_1} \right) = - \left[2 + \frac{a - c}{b}\right] < 0, \quad \frac{d}{dq_2} \left( \frac{d\pi_2}{dq_2} \right) = - \left[2 + \frac{a - c}{4b}\right] < 0
\]

hence \((q^*_1, q^*_2)\) is a local equilibrium. The values of the conjectural variation functions are respectively

\[
\left( \frac{dq_2}{dq_1} \right)^C = -\frac{1}{2} \quad \text{and} \quad \left( \frac{dq_1}{dq_2} \right)^C = 0
\]

(15)

To verify that the equilibrium is indeed consistent we have to verify that (8) are satisfied.

Differentiating firm \#1 first order condition gives:

\[
\left[ - \left(2 + \frac{c - a}{2b}\right) - 2q_1 + \frac{2b}{a - c} q_2 \right] dq_1 - \left[1 + \frac{2b}{c - a}\right] dq_2 = 0
\]

so the first firm’s reaction to output variations of firm \#2, is, at \((q^*_1, q^*_2)\),

\[
\frac{dq_1}{dq_2} = \frac{0}{3 + \frac{a - c}{2b}} = 0
\]
and the second firm’s conjecture is therefore consistent. Similarly from differentiating firm #2 first order condition, it comes:

\[-(2 + \frac{c}{2b}) + 2q_2 + \frac{1}{2}q_1 dq_2 - \left[1 + \frac{1}{2}q_2\right] dq_1 = 0\]

so that the second firm’s reaction to output changes by firm #1 is given, at \((q_1^*, q_2^*)\), by:

\[\frac{dq_2}{dq_1} = \frac{1 + \frac{a - c}{8b}}{2 \left[1 + \frac{a - c}{8b}\right]} = -\frac{1}{2}\]

It follows that firm #1 conjecture is also correct.

But \(\left(\frac{a - c}{2b}, \frac{a - c}{4b}\right)\) are the output levels of a Stackelberg equilibrium with #1 as leader and #2 as follower. As in the Cournot case many other conjectural variation functions could provide the same result.

With Stackelberg we got \(\frac{3}{8}\) of the competitive output and with Cournot \(\frac{3}{5}\). Other situations nearer the two extreme ones, competition or collusion, could also be obtained as rational conjectural equilibria. Example 3 shows how \(\frac{5}{8}\) of the competitive output could appear.

Example 3: The opposite weak Stackelberg case

Suppose that the firms in this case hold conjectural variations respectively equal to \(\frac{1}{2}\) for firm #1 and 0 for firm #2, from which the above name was given. For such conjectures, the equilibrium is

\[q_1^* = \frac{a - c}{4b} \quad \text{and} \quad q_2^* = \frac{3}{8}\left(\frac{a - c}{b}\right)\]

giving a total output of

\[\frac{5}{8}\left(\frac{a - c}{b}\right)\].

From the reaction functions of the duopolists, one can verify that the slopes are

\[\frac{dq_2}{dq_1} = -\frac{1}{2}\]
\[\frac{dq_1}{dq_2} = -\frac{2}{5}\]

which are different from the conjectures and so this equilibrium is also inconsistent.
Now consider the following.
Suppose the conjectural variation functions are:

\[
\begin{align*}
\text{Firm } \#1 & \quad \left( \frac{dq_2}{dq_1} \right)^c = \frac{7}{4} + \frac{b}{a-c} q_1 + \frac{4b}{c-a} q_2 \\
\text{Firm } \#2 & \quad \left( \frac{dq_1}{dq_2} \right)^c = \frac{10}{12} + \frac{2b}{a-c} q_2 + \frac{19b}{3(c-a)} q_1
\end{align*}
\] (16)

The system of first order conditions is:

\[
\begin{align*}
\frac{d\pi_1}{dq_1} &= \frac{a-c}{b} - \frac{15}{4} q_1 - q_2 - \frac{b}{a-c} q_1^2 + \frac{4b}{a-c} q_1 q_2 = 0 \\
\frac{d\pi_2}{dq_2} &= \frac{a-c}{b} - \frac{34}{12} q_2 - q_1 - \frac{2b}{a-c} q_2^2 + \frac{19b}{3(a-c)} q_1 q_2 = 0
\end{align*}
\]

A solution is \((q_1^*, q_2^*) = \left( \frac{a-c}{4b}, \frac{3(a-c)}{8b} \right)\). For \((q_1^*, q_2^*)\):

\[
\begin{align*}
\frac{d}{dq_1} \left( \frac{d\pi_1}{dq_1} \right) &= -\frac{11}{4} < 0, & \frac{d}{dq_2} \left( \frac{d\pi_2}{dq_2} \right) &= -\frac{33}{12} < 0
\end{align*}
\]

hence \((q_1^*, q_2^*)\) is a local equilibrium. And the values of the conjectural variation functions at the equilibrium are respectively

\[
\left( \frac{dq_2}{dq_1} \right)_{q^*} = \frac{1}{2} \left( \frac{dq_1}{dq_2} \right)_{q^*} = 0.
\]

Following the now well established procedure, i.e., differentiating the first order conditions, it comes for firm \#1:

\[
- \left[ \frac{15}{4} - \frac{2b}{a-c} q_1 + \frac{4b}{a-c} q_2 \right] dq_1 - \left[ 1 + \frac{4b}{c-a} q_1 \right] dq_2 = 0
\]

and the slope of firm \#1’s reaction function is, at \((q_1^*, q_2^*)\):

\[
\frac{dq_1}{dq_2} = \frac{0}{-11/4} = 0.
\]

The conjecture of firm 2 is therefore consistent. For firm \#2,

\[
- \left[ \frac{34}{12} - \frac{4b}{a-c} q_2 + \frac{19b}{a-c} q_1 \right] dq_1 - \left[ 1 + \frac{19b}{3(c-a)} q_1 \right] dq_2 = 0
\]

and so the slope of firm \#2’s reaction function at \((q_1^*, q_2^*)\) is:

\[
\frac{dq_2}{dq_1} = -\frac{11/8}{-33/12} = \frac{1}{2}.
\]
Examples could be multiplied. In the next section we present the algorithm by which one can compute the parameters of the conjectural variation functions making any sustainable output pair a rational conjectural equilibrium.

III. THE ALGORITHM

Although the examples of section II used linear demand and cost functions, the algorithm presented here uses general demand and cost functions. It is only required that they be twice differentiable. Also, the starting production levels must be sustainable, i.e., profits should be non-negative. Moreover, we need for the algorithm that $P'(Q) \neq 0$ and also that

$$[C_1(q_1) - P - q_1 P'][C_2(q_2) - P - q_2 P'] \neq (P')^2 q_1 q_2$$

which is sufficient to make $F_1 F_2 \neq 1$ in the algorithm. Condition (17) rules out both the competitive equilibrium and the collusive equilibrium. If $B_i > 0$ for $i = 1, 2$ in step 3 of the algorithm, then one can drop condition (17); it is therefore not necessary for the existence of the desired conjectural variation functions.

We can now give the algorithm in 6 steps.

**Step 1:** Given $(q_1, q_2) > 0, P(Q), C_1(q_1), C_2(q_2)$ compute $P, P', P'', C_1', C_1'', C_2', C_2''$ and check that $P(Q)q_i \geq C_i(q_i)$ for $i = 1, 2$.

**Step 2:** Check if the following conditions are met

$$[C_1 - P - q_1 P'][C_2 - P - q_2 P'] \neq (P')^2 q_1 q_2.$$

If not, the algorithm may not work.

**Step 3:** Compute sequentially

$$F_1 = [C_1 - P - q_1 P']/[q_1 P']$$

$$F_2 = [C_2 - P - q_2 P']/[q_2 P']$$

$$A_i = (q_i P')^{-1}[F_i]2 P' F_i + q_i P'' + q_i P' F_i - C_i']$$

$$+ P' + q_i P'[1 + F_i]$$

for $i = 1, 2$.

$$B_i = C_i'' - 2 P'[1 + F_i] - q_i P''[1 + F_i]^2 + q_i P' F_i A_i$$

for $i = 1, 2$.

$$D_i = q_i P'[1 - F_i F_2]$$

for $i = 1, 2$. 
\[ E_i = \left[ -P'(2 + F_i) - q_i P'(1 + F_i) + C_i/(q_i) \right] \]

for \( i = 1, 2 \).

\textbf{Step 4:} Choose any \( \beta_1, \beta_2 \) such that, for \( i = 1, 2 \)
\[ D_i \beta_i < B_i \text{ and } \beta_i \neq E_i. \]

\textbf{Step 5:} Compute
\[
\begin{align*}
\gamma_i &= -\beta_i F_j - A_i \\
\alpha_i &= -\beta_i q_i - \gamma_i q_j + F_i \\
\end{align*}
\]
for \( i = 1, 2 \) and \( j \neq i \).

\textbf{Step 6:} Write the conjectural variation functions
\[
\left( \frac{\partial q_j}{\partial q_i} \right)^C = \alpha_i + \beta_i q_i + \gamma_i q_j \quad i, j = 1, 2 \quad \text{and} \quad i \neq j
\]
which will make the pair \((q_1, q_2)\) we started with a rational conjectural equilibrium.

The formal theorems and proofs on which the algorithm is based can be found in Boyer and Moreaux [2].

It is clear from step 4 that there exists a whole class of conjectural variation functions, one for each value of \((\beta_1, \beta_2)\). Each pair of functions so obtained will give the same reaction function, they will lead to the same solution and the equilibrium will always be consistent.

\section*{IV. CONCLUSION}

We have illustrated in this paper that any pair of production levels \((q_1, q_2)\) which might or might not be the equilibrium solution to some duopoly model with constant conjectural variations can be obtained as a rational conjectural equilibrium for appropriately defined conjectural variation functions. We gave 3 examples and also a general procedure to generate such rational conjectural equilibria.

We also observed that, for the three examples given, a reinterpretation of the constant conjectural variation model was possible. For the case of the Cournot duopoly model with zero conjectural variations we noted that although this model is inconsistent—since at the equilibrium the conjectures are not validated by the best reply or reaction strategy of the duopolists—it is the case that the \textit{values} of the duopolists’ conjectural variation functions—which make the Cournot’s quantities a rational conjectural equilibrium—are both 0 at the equilibrium. Therefore, the zero conjectural variation model of Cournot is in a sense a rational conjectural equilibrium since there exists a pair of conjectural variation functions making the
Cournot solution a rational conjectural equilibrium and whose values at the equilibrium are 0. Therefore, it is correct for the duopolists to use 0 as conjectural variations at the Cournot equilibrium. But in differentiating the necessary condition for profit maximization to find the slope of the reaction function, one must be careful not to consider that the conjectures are zero identically. In other words, by properly differentiating Cournot's reaction function it turns out that the slope of the reaction function is effectively 0 at the equilibrium. In that sense, Cournot's duopoly model is consistent and rational. A similar analysis holds for the other constant conjectures models.

It is worth noting also that restricting our attention to conjectural variation functions of the linear form (6) has no cost in terms of generality. Every model of duopoly based on conjectural variations can be rationalized through appropriately defined linear conjectural functions.

These results suggest that the requirement of consistency in conjectural models of duopoly allows a reinterpretation in terms of local conjectures which makes the usual constant conjectures consistent. It does not however restrict the duopoly solutions which can be theoretically sound. The restrictions obtained by Bresnahan, Perry, Kamien and Schwartz, Ulph and the others come from somewhat misspecified models whose basic elements are difficult to justify on rational grounds. It seems to us that the research effort should now deal with the origins, development and dynamics of the conjectures themselves to lift the arbitrariness which the consistency requirement has not eliminated.

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