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Uniqueness of Cournot Equilibrium: New Results From Old Methods

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This paper provides sufficient conditions for the existence of a unique Cournot equilibrium. Previous uniqueness results have depended on an assumption of non-degeneracy of equilibrium. As we illustrate, this assumption often fails in multi-stage games with proper Cournot subgames. Since our uniqueness results do not depend on this assumption, they are more widely applicable.

1. INTRODUCTION

This paper provides conditions sufficient for the existence of a unique pure-strategy equilibrium in simultaneous-move games of quantity competition where firms sell perfect substitutes. Its practical contribution is to provide conditions which are less restrictive than previous papers while retaining the elementary methods and constructive aspect of Szidarovsky and Yakowitz (1977).

Szidarovsky and Yakowitz (1977) restrict attention to concave demand. Kolstad and Mathiesen (1987) relax this restriction but instead require the absence of degenerate equilibria. They define a degenerate equilibrium as one where some firm produces zero output but has a marginal cost equal to the equilibrium price; therefore, such a firm is just on the margin of producing. When a degenerate equilibrium exists, Kolstad and Mathiesen (1987) say nothing about the number of non-degenerate (or degenerate) equilibria.

At first glance, this restriction may seem rather mild. For, as Kolstad and Mathiesen point out (p. 683), if a degenerate equilibrium exists in a Cournot game, a small perturbation of the exogenous cost and demand parameters will eliminate it. Hence, degenerate equilibria are unlikely to occur in one-stage games in a measure-theoretic sense. On the other hand, in a given Cournot game, a degenerate equilibrium may be known to exist. In such circumstances, Kolstad and Mathiesen's theorem cannot be used to establish that it is the only equilibrium.

One-stage games, however, are not the only context in which such a uniqueness theorem might be useful. In recent years, economists have been led increasingly to analyze the subgame-perfect equilibria of multi-stage games with Cournot games (simultaneous moves in quantities) as proper subgames. One or more of those proper subgames may contain a degenerate equilibrium. Even if only one subgame in a continuum contains a degenerate equilibrium, the equilibria of that subgame must then be analyzed in determining the set of subgame perfect equilibria. This is true whether or not the degenerate equilibrium to the subgame arises in the actual play of the game.

When it does arise in the play of the game, perturbing the set of parameters exogenous to the game *as a whole* will not necessarily re-establish a non-degenerate equilibrium. For in a multi-stage game there will be *additional* "parameters" which are exogenous to the subgame but endogenous to the game as a whole. Typically, those "parameters" are chosen strategically in previous stages of the game by players who seek to alter the equilibria in subsequent subgames. It thus becomes irrelevant that the parameters exogenous to the subgame would be unlikely to result in degenerate equilibria if *all* of them were chosen by chance. For they are not. Some are instead chosen by motivated players, who will adjust their actions in response to any perturbation. We illustrate these points in the Appendix. In our example, the unique degenerate equilibrium to one of the proper Cournot subgames *always* arises in the actual play of the game.[1]

To summarize, in multi-stage games degenerate equilibria may occur in some subgames and, indeed, may arise in the play of the game. Moreover, such degeneracy may survive perturbations of the cost and demand parameters. In the presence of degenerate equilibria, the theorem of Kolstad and Mathiesen is inapplicable while the usefulness of our theorem is undiminished.

2. NECESSARY AND SUFFICIENT CONDITIONS

Consider an industry composed of N firms producing a homogeneous good. Firm i produces the good in quantity $q_i \geqq 0$. Its cost function is $C_i(q_i)$, defined for all $q_i \geqq 0$. The inverse demand function for the good is $p(Q)$, where $Q = \sum_{i=1}^{N} q_i$ is the total industry output. Assume that:

Assumption 1. There exists a $\xi \in (0, \infty)$ such that $p(Q) > 0$ for $Q \in [0, \xi)$ and $p(Q) = 0$ for $Q \in [\xi, \infty)$.

Assumption 2. $p(Q)$ is twice-continuously differentiable and $p'(Q) \leqq 0$ for $Q \in [0, \xi)$.

Assumption 3. $C_i(q_i)$ is twice-continuously differentiable and, for any $q_i > 0$, $C_i'(q_i) > (0)$, $i = 1, \ldots, N$.

Assumption 4. For all $Q \in [0, \xi)$ and $i = 1, \ldots, N$, there exists some $\alpha < 0$ (possibly dependent on Q and i) such that $p'(Q) - C_i''(q_i) \leqq \alpha < 0$ for every $q_i \geqq 0$.

For any cost function satisfying Assumptions 3 and 4, define the extended function, $\Gamma_i(x_i)$, as follows:

$$\Gamma_i(x_i) = \begin{cases} C_i(x_i) & \text{if } x_i \geqq 0 \\ C_i(0) + C_i'(0)x_i + \frac{1}{2}C_i''(0)x_i^2 & \text{if } x_i < 0. \end{cases}$$

Note that $\Gamma_i(x_i)$ is twice-continuously differentiable for any $x_i \in (-\infty, \infty)$. Moreover, since $\Gamma_i''(x_i) = C_i''(0)$ for $x_i < 0$ and Assumption 4 holds for $q_i \geqq 0$, then for all $Q \in [0, \xi)$ and $x_i \in (-\infty, \infty)$, $i = 1, \ldots, N$, there exists some $\alpha < 0$ (possibly dependent on Q and i) such that $p'(Q) - \Gamma_i''(x_i) \leqq \infty$. Thus the appropriately modified Assumptions 3 and 4 must hold for the extended function.

1. We are grateful to an anonymous referee for suggesting this example.

Define $g_i(x_i, Z) = p(Z) + x_i p'(Z) - \Gamma_i'(x_i)$, where Z is some exogenous parameter. Then Cournot equilibria must satisfy:

$$g_i(q_i, Z) \leqq 0, \qquad q_i \geqq 0, \qquad g_i(q_i, Z)q_i = 0, \qquad i = 1, \ldots, N \tag{1}$$

and

$$Z = Q. \tag{2}$$

We proceed in three steps. First, we demonstrate that there exists at least one solution to (1) and (2). Second, we provide a condition for the solution of (1) and (2) to be unique. Third, we relate the solutions of (1) and (2) to the set of Cournot equilibria. Since any Cournot equilibrium solves (1) and (2), uniqueness of the solution of (1) and (2) ensures that there is *at most* one Cournot equilibrium. We conclude by providing conditions which ensure that any solution to (1) and (2) is a Cournot equilibrium; if the solution to (1) and (2) is unique, therefore, the Cournot equilibrium is unique.

It is straightforward to verify that there exists at least one solution to (1) and (2). Consider first condition (1). By Assumption 1 through Assumption 3, the function $g_i(x_i, Z)$ has continuous partial derivatives for all $Z \in [0, \xi)$ and $x_i \in (-\infty, \infty)$. By Assumption 4, its partial derivative with respect to x_i is negative, bounded away from zero. There must therefore exist a unique $x_i(Z)$ such that $g_i(x_i(Z), Z) = 0$. This implicit solution being unique, it must be continuous at all $Z \in [0, \xi)$. For suppose it is discontinuous at some $Z^0 \in [0, \xi)$. Then, in a neighbourhood of Z^0, there exists no continuous solution to $g_i(x_i, Z) = 0$ since $x_i(Z^0)$ is unique. But this violates the implicit function theorem at that point. It follows that condition (1) has a unique solution for all $Z \in [0, \xi)$, given by $q_i(Z) = \text{Max}\,(0, x_i(Z))$.

Now since $q_i(Z)$ is continuous for any $Z \in [0, \xi)$, so is $Q(Z) = \sum_{i=1}^{n} q_i(Z)$. We also know that $Q(0) \geqq 0$, since $q_i \geqq 0$ for all $i = 1, \ldots, N$. Furthermore, $\lim_{Z\uparrow\xi} q_i(Z) = 0$ by Assumption 1–Assumption 3. There must therefore exist at least one $Z^E \in [0, \xi)$ which solves $Q(Z^E) = Z^E$ and the corresponding $q_i^E(Z^E)$, $i = 1, \ldots, N$, constitutes a solution to (1) and (2). Since $Q(Z) = 0$ for $Z \geqq \xi$, there can be no solutions to (1) and (2) with $Z \geqq \xi$.

We now provide a condition for the solution of (1) and (2) to be unique:

Theorem. *Suppose that Assumptions 1 through 4 hold. Then, if (only if) at all* q_i^E, $i = 1, \ldots, N$, *we have*

$$\sum_{i\in M(Q^E)} \frac{p'(Q^E) + q_i^E p''(Q^E)}{C_i''(q_i^E) - p'(Q^E)} < 1 \ (\leqq 1) \tag{3}$$

where $M(Q^E) = \{i \mid q_i(Q^E) > 0\}$, *there exists exactly one solution to* (1) *and* (2).

Proof. Let $q_i'(Z)^-$ and $q_i'(Z)^+$ denote the left-hand and right-hand derivatives of $q_i(Z)$. The initial step of the proof is to establish that:

(i) $q_i'(Z)^+ \geqq q_i'(Z)^-$

(ii) $q_i'(Z)^+ = \begin{cases} 0 & \text{if } q_i(Z) = 0 \\ x_i'(Z) & \text{if } q_i(Z) > 0. \end{cases}$

Since $x_i(Z)$ is unique for all $Z \in [0, \xi)$ and given Assumption 4, we can again invoke the implicit function theorem to show that $x_i(Z)$ has a continuous derivative for $Z \in [0, \xi)$ and it is given by:

$$x_i'(Z) = \frac{p'(Z) + x_i(Z)p''(Z)}{C_i''(x_i(Z)) - p'(Z)}.$$

If $x_i(Z)>0$ then $q_i(Z)=x_i(Z)>0$ and $q_i'(Z)^+=q_i'(Z)^-=x_i'(Z)$. Thus (i) and (ii) both hold. If instead $x_i(Z')<0$ for some $Z'\in[0,\xi)$ then $q_i(Z)=0$ at Z' and in a neighbourhood of Z', since $x_i(Z)$ is continuous at Z' and hence $x_i(Z)<0$ in some neighbourhood of Z'. Therefore $q_i(Z')^+=q_i(Z')^-=0$ and (i) and (ii) hold. Finally, if $x_i(Z')=0$ for some $Z'\in[0,\xi)$ then $q_i(Z)=x_i(Z')=0$. By continuity and nonnegativity of $q_i(Z)$, $q_i(Z)^-\leqq 0$ and $q_i(Z)^+\leqq 0$. But Assumptions 2 and 4 imply $x_i'(Z)\leqq 0$ when $x_i(Z)=0$; $q_i'(Z')^+>0$ is therefore impossible. Hence $q_i'(Z')^-\leqq q_i'(Z')^+=0$ and again (i) and (ii) hold.

Now let $Q'(Z)^+$ denote the right-hand derivative of $Q(Z)$. Then $Q'(Z)^+=\sum_{i=1}^N q_i'(Z)^+=\sum_{i\in M(Z)} q_i'(Z)^+=\sum_{i\in M(Z)} x_i'(Z)$. The first equality follows from the definition of $Q(Z)$, while the second and third follow from (ii). Moreover, (i) implies that $Q'(Z)^-\leqq Q'(Z)^+$, and an upper bound on $Q'(Z)^+$ also bounds $Q'(Z)^-$. It follows that if (only if) we have $Q'(Z^E)^+<1(\leqq 1)$, there must be only one $Z^E(=Q^E)$ and hence only one $q_i^E(Z^E)$, $i=1,\dots,N$. But:

$$Q'(Z^E)^+=\sum_{i\in M(Z^E)} x_i'(Z^E)^+=\sum_{i\in M(Z^E)}\frac{p'(Z^E)+q_i(Z^E)p''(Z^E)}{C_i''(q_i(Z^E))-p'(Z^E)}$$

and the condition stated in the theorem follows directly.[2] ||

Since any Cournot equilibrium must satisfy (1) and (2), Assumptions 1 through 4 and (3) therefore ensure that there exists at most one Cournot equilibrium. They also identify the solution to (1) and (2) as the only candidate for the Cournot equilibrium.[3] If, in addition, conditions (1) and (2) are known to be sufficient for a Cournot equilibrium, then Assumptions 1 through 4 and (3) ensure that there exists exactly one Cournot equilibrium.

Notice that if we require, in addition to Assumptions 1 through 4, that a firm's marginal revenue be a non-increasing function of the output of its rivals, so that:

Assumption 5. $p'(Q)+q_ip''(Q)\leqq 0$ for all $Q\in[0,\xi)$, $q_i\in[0,Q]$, $i=1,\dots,N$

then condition (3) holds with strict inequality. Assumptions 4 and 5 combined also ensure that each firm's profit function is strictly concave in its own output. Thus Assumptions 1 through 5 ensure existence of a unique Cournot equilibrium[4].

Assumption 5 is frequently involved in discussions of existence of Cournot equilibrium (see in particular Novshek (1985) and Shapiro (1989)). Given that $p'(Q)\leqq 0$ by Assumption 2, it in fact is equivalent to the Novshek assumption that $p'(Q)+Qp''(Q)\leqq 0$ for $Q\in[0,\xi)$. However, this assumption is unnecessarily strong for existence of equilibrium, given Assumptions 1 through 4. For, any solution to (1) and (2) is a Cournot equilibrium if and only if q_i^E yields a global maximum of $p(q_i+Q_{-i}^E)q_i-C_i(q_i)$, where $Q_{-i}^E=Q^E-q_i$ and $i=1,\dots,N$. As long as this condition holds at some solution to (1) and (2), existence of a Cournot equilibrium is assured. It is unnecessary for existence

2. Given Assumption 1 through 4 and (3), we will necessarily have $Z^E\in[0,\xi]$ and $Q(Z)-Z\gtreqqless 0$ for all $Z\lesseqqgtr Z^E$, and the method of interval bisection (see for example Conrad and Clark (1987), p. 40) would in such a case rapidly converge to the unique solution to (1) and (2).

3. Not every solution to (1) and (2) need be a Cournot equilibrium. Although (1) and (2) ensure that each firm selects a point in reply to the outputs of the other firms which satisfies the necessary conditions for a local maximum of its profit function, this need not be a local best reply, much less a global best reply, for every firm.

4. If firms have identical cost functions and the equilibrium is unique, then Assumptions 1 through 4 imply that every firm must produce the identical output. For Z^E is unique by hypothesis, Assumptions 1 through 4 imply that $q_i(Z^E)$ is unique for each i, and (1) implies that the function $q_i(\cdot)$ is the same for each i.

of Cournot equilibrium to assume that this condition holds at every solution to (1) and (2). Moreover, at the designated solution to (1) and (2), it is unnecessary that q_i^E be globally optimal in response to every aggregate output of the firms, but merely to Q_i^E. Finally, it is unnecessary that each firm's profit function be strictly concave (or even pseudo-concave) everywhere.

3. CONCLUSION

The uniqueness conditions (3) are the same as those derived by Kolstad and Mathiesen (1987) (equation (16), Corollary 3.1, p. 687). However their theorem has limited applicability when non-degenerate equilibria cannot be ruled out. Since its proof relies on the non-degeneracy of Cournot equilibria, it does not exclude the possibility that there exists one or more degenerate Cournot equilibria (i.e. equilibria where $q_i = 0$ and $g_i(q_i, Q) = 0$ for some or all $i \notin M(Q)$). Furthermore, and maybe more important, when there is a degenerate equilibrium, it can say nothing about the number of other equilibria, degenerate or not. Since our proof is not restricted to the class of non-degenerate Cournot equilibria, the uniqueness conditions of our theorem apply to any Cournot equilibrium, degenerate or not. Hence our results fill an important gap.

Our approach is closest to that of Szidarovsky and Yakowitz (1977). They use it to show the existence of a unique Cournot equilibrium when marginal cost is increasing and inverse demand is downward sloping and concave. Their assumptions imply Assumptions 1 through 5, but are, of course, unnecessarily strong.

APPENDIX

In this appendix, we consider a two-stage game in which each of a continuum of proper subgames involves Cournot competition. To investigate the existence of a unique pure-strategy subgame-perfect equilibrium of this game, it is necessary to verify that the Cournot equilibrium of each proper subgame is unique.

In the example we propose, only one subgame among the continuum involves a degenerate equilibrium. Even this is sufficient, however, for Kolstad and Mathiesen's results to be inapplicable. However, our results readily apply. They establish that the equilibrium in each subgame is unique.

In the example, the unique optimal strategy for the "leader" in the first stage ensures that the subgame with the degenerate Cournot equilibrium is played in the second stage. Indeed, even if the exogenous parameters are perturbed slightly, the leader will alter his strategy so that, despite the perturbation, the subgame with the degenerate equilibrium will *still* be played in the second stage.

Consider the following two-stage limit-pricing game. Suppose one player (the "leader") precommits to an output level (denoted q_0) and each of n Cournot "followers" observes this choice and then simultaneously selects its own output (denoted q_i, $i = 1, 2, \dots, n$). Assume that each follower has the same positive constant marginal cost, denoted c; for simplicity, assume the leader produces costlessly. Denote the inverse demand curve by $P = a - \sum_{i=0}^{n} q_i$.

It is straightforward to verify that

$$q_i = \begin{cases} (a - q_0 - c)/(n+1) & \text{if } q_0 \leqq a - c \\ 0 & \text{if } q_0 > a - c \end{cases}$$

is one Nash equilibrium strategy combination in the proper subgame indexed by q_0. Hence there is a degenerate equilibrium in the subgame with $q_0 = a - c$.

Given the existence of one subgame with a degenerate equilibrium, Kolstad and Mathiesen's results cannot be used to determine how many other equilibria (degenerate or non-degenerate) exist in that subgame. But until the payoff associated with *each* of the equilibria in that subgame is known, we cannot determine how the leader will move in the previous stage, whether in fact there will be multiple equilibria, and if so whether the leader's move will be different in these different equilibria. Fortunately, our Assumptions 1 through 5 hold in each subgame; therefore, the output combination specified above for each subgame constitutes its unique equilibrium.

The payoff of the leader can therefore be written as a single-valued function of q_0:

$$\pi_0\begin{cases} q_0(a-q_0-[n/(n+1)](a-q_0-c)) & \text{if } q_0 \leqq a-c \\ q_0(a-q_0) & \text{if } q_0 > a-c. \end{cases}$$

π_0 is continuous, strictly concave, and kinked at $q_0 = a - c$. Differentiating, we note that

$$\pi_0' = \begin{cases} a-q_0-[n/(n+1)](a-q_0-c)+q_0((n/n+1)-1) & \text{if } q_0 < a-c \\ \text{undefined} & \text{if } q_0 = a-c \\ a-2q_0 & \text{if } q_0 > a-c \end{cases}$$

The unique maximum occurs at $q_0 = a - c$ if and only if at that point the right-hand derivative is negative and the left-hand derivative is positive. Evaluating the left- and right-hand derivatives at $q_0 = a - c$, we conclude that the leader will strictly prefer the degenerate equilibrium in the subgame (i.e. will set $q_0 = a - c$) if and only if $0 < a/(n+2) < c < a/2$. To illustrate: if $a = 100$ and $n = 48$, then for any $2 < c < 50$, the leader would choose his output so that the subgame with the degenerate equilibrium would be played.

Notice that a perturbation of c and a will not ensure that the equilibrium in the subgame chosen by the leader is non-degenerate. Indeed, he will respond to a small perturbation by adjusting his output to the new value of $(a - c)$ as long as the perturbed cost lies in the perturbed open interval. Hence, a degenerate equilibrium will still occur.

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